The Geometric Cone Relations of Polyhedra

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Let P be a d-dimensional polytope of the Euclidean space $\mathbf{R}^n$. For each $i = -1, 0, \cdots, d-1$, let $f_j(P)$ denote the number of j-faces of P. The empty set $\emptyset$, usually denoted $\hat{0}$, is the unique face of P of dimension -1. It is well known that the Dehn-Sommervile equations

$$f_i(P) = \sum_{j=i}^{d-1} (-1)^{d-1-j} \binom{j+1}{i+1} f_j(P), i = -1, \cdots, d-1 \tag{1}$$

are the only linear relations on the f-vectors $(f_{-1}, f_0, f_1, \cdots, f_{d-1})$ of convex polytopes [B]. These relations are naturally generalized by using cone functions in [Ch]. The numerical form of these cone relations by taking certain integration over the unit ball of $\langle P \rangle$ are the angle-sum relations

$$f_i(P) - \alpha_i(P) = \sum_{j=i}^{d-1} (-1)^{d-1-j} \binom{j+1}{i+1} \alpha_j(P), i = -1, \cdots, d-1 \tag{2}$$

The purpose of this note is to generalize these relations to simplicial complexes and cubical complexes both in cone functions and angles.

Throughout we denote by $\mathbf{V}$ a finite dimensional vector space over an ordered field $\mathbf{F}$ with an inner product $\langle , \rangle$. The topology of $\mathbf{V}$ is that of generated by the order topology of the ordered field $\mathbf{F}$. The *indicator* function of a subset E of $\mathbf{V}$ is the characteristic function 1_E on $\mathbf{V}$, i.e., $1_E(x) = 1$ for $x \in E$ and $1_E(x) = 0$ otherwise. By a *polyhedra* we mean a subset of $\mathbf{V}$ which can be obtained by taking unions, intersections, and complements

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finitely many times of open half-spaces $\{x \in \mathbf{V} | \varphi(x) < 0\}$, where φ is a linear function on $\mathbf{V}$. The interior of a convex polyhedron in the affine subspace that it spans is called a *relatively open* convex polyhedron. We use $\mathcal{P}$ to denote the class of all relatively open convex polyhedra.

Let P be a relatively open polytope of $\mathbf{V}$. A relatively open polytope F is said to be a *face* of P if the closure $\bar{F}$ of F is a face of $\bar{P}$ in the ordinary sense, and this is denoted by $F \leq P$ or $P \geq F$. We shall consider the *interior* cone $C(F,P)$ and *exterior* cone $C^*(F,P)$ of P near its face F, which are defined by

$$C(F,P) = \{v \in \mathbf{V} \mid \exists x \in F, y \in P, t > 0 \text{ s.t. } tv = y - x\},$$

$$C^*(F,P) = \{v \in V \mid \langle u, v\rangle \leq 0, \forall u \in C(F,P)\}$$

Note that when $F = P$, $C(F,F)$ is a vector subspace of dimension $\dim F$, and is denoted $\langle F\rangle$. The *intrinsic* interior cone $C^\wedge(F,P)$ in [B, M2] and exterior cone $C^\vee(F,P)$ of P near its face F are also considered and they are defined by

$$C^\wedge(F,P) = \langle F\rangle^\perp \cap C(F,P),$$

$$C^\vee(F,P) = C^*(F,P) \cap \langle P\rangle.$$

We denote the indicator functions of the cones $C(F,P)$, $C^*(F,P)$, $C^\wedge(F,P)$, and $\mathrm{ri}(C^\vee(F,P))$ by $T(F,P)$, $K(F,P)$, $A(F,P)$, and $B(F,P)$, respectively. These functions on $\mathbf{V}$ are the elements of the Minkowski algebra $S(\mathbf{V},\mathcal{P})$ which is the vector space generated by the indicator functions of members of $\mathcal{P}$, and the multiplication is the convolution induced by the vector addition of $\mathbf{V}$. There is a linear functional χ, called the *Euler characteristic*, on $S(\mathbf{V},\mathcal{P})$ such that $\chi(1_P) = (-1)^{\dim P}$ for each relatively open convex polyhedron. There are three useful operators, *reflection* $^-$, *closure* $\bar{}$, and *dual* * on $S(\mathbf{V},\mathcal{P})$, defined by

$$f^-(x) = \chi(f \cdot 1_{\{-x\}}), \forall x \in \mathbf{V},$$

$$\bar{f}(x) = \lim_{r\to 0} \chi(f \cdot 1_{B(x,r)}), \forall x \in \mathbf{V},$$

$$f^*(x) = \chi(f \cdot 1_{\{v \in \mathbf{V} | \langle x,v\rangle \leq 0\}}), \forall x \in \mathbf{V},$$

where $B(x,r)$ is the closed ball of radius r centered at x.

Let X be a polyhedron of $\mathbf{V}$. A *regular decomposition* of X is a collection $\mathcal{D}$ of disjoint relatively open convex polyhedra such that $X = \bigcup_{P\in\mathcal{D}}$ and the collection $\mathcal{F}(X,\mathcal{D}) = \{G | G \leq P \in \mathcal{D}\}$ is also disjoint. The set $\mathcal{F} = \mathcal{F}(X,\mathcal{D})$ is called the *face system* of X with respect to the decomposition $\mathcal{D}$. Note that X is closed if and only if $\mathcal{F}(X,\mathcal{D}) = \mathcal{D}$. In fact, $\mathcal{F}$ is a regular decomposition of $\bar{X}$, the closure of X. The cone $C(F,X)$ of X near a face $F \in \mathcal{F}(X,\mathcal{D})$ is the disjoint union of $C(F,\mathcal{D}) = \{C(F,G) | F \leq G \in \mathcal{D}\}$. $C(F,X)$ is a closed cone if and only if $F \in \mathcal{D}$. The face system of $C(F,\mathcal{D})$ is isomorphic to the face poset $\check{F} = \{G \in \mathcal{F} | F \leq G\}$ of X near F. Let f be an incidence function with values in the commutative ring R, i.e., $f(F,P) = 0$ if F is not a face of P, we associate with f another incidence function f', defined by

$$f'(F,P) = (-1)^{\dim P - \dim F} f(F,P), \forall F \leq P.$$

For each face $F \in \mathcal{F}(X,\mathcal{D})$ and $j = 0, 1, \cdots, \dim X$, we define

$$f(F,X) = \sum_{F \leq G \in \mathcal{D}} f(F,G),$$

$$f_j(X) = \sum_{F \in \mathcal{F}, \dim F = j} f(F,X).$$

For example, if f is the function T, then $T(F,X)$ is the characteristic function of the tangent cone of X near F, and the $T_j(X)$ is the sum of the characteristic functions of tangent cones of X near its all j-dimensional faces.

Proposition 1 *Let X be a polyhedron with a regular decomposition $\mathcal{D}$. Then*

$$\sum_{F \leq G \in \mathcal{F}} (-1)^{\dim G} T(G,X) = \bar{T}(-F,-X) = \bar{T}^{-}(F,X) \tag{3}$$

Proof For each pair (G,P) of relatively open faces such that $F \leq G \leq P \in \mathcal{D}$, it claer that $C(G,P) = C(C(F,G),C(F,P))$. Then $C(G,X) = C(C(F,G),C(F,X))$. So the left side of (**??**) can be written as

$$\sum_{C(F,F) \leq C(F,G) \in C(F,\mathcal{F})} (-1)^{\dim C(F,G)} T(C(F,G),C(F,X)).$$

With the generalized Gram-Sommerville theorem [Ch1] and the definition of the cone near ∞, the left side of (**??**) is then equal to

$$\chi(C(F,X))1_{\{o\}} - \sum_{C(F,F)\le C(F,P)\in C(F,\mathcal{D})} T_{C(F,P)}(\infty).$$

Note that $C(F,\mathcal{F})$ is in one-to-one correspondent with $\check{F}$ and $\dim C(F,G) = \dim G$ for every $G\in\check{F}$. We have

$$\chi(\check{F})1_{\{o\}} - \sum_{F\le P\in\mathcal{D}} \left[(-1)^{\dim P} - (-1)^{\dim P} 1_{C(\infty, C(F,P))}\right].$$

Since $C(F,P)$ are convex cones, then $C(\infty, C(F,P)) = -\mathrm{cl}[C(F,P)]$. So the left side of (**??**) becomes

$$\sum_{F\le P\in\mathcal{D}} (-1)^{\dim P} 1_{-\mathrm{cl}[C(F,P)]} = \sum_{F\le P\in\mathcal{D}} \bar{T}(-F,-P) = \bar{T}(-F,-\check{F}).$$

Q.E.D.

Now we take the sum on both sides of (**??**) over all i-dimensional faces of X, the left side can be written as

$$\begin{aligned} &\sum_{G\in\mathcal{F},\dim G\ge i} (-1)^{\dim G} \sum_{F\le G,\dim F=i} T(G,X) \\ &= \sum_{j=i}^{\dim X} \sum_{G\in\mathcal{F},\dim G=j} (-1)^j \binom{j+1}{i+1} T(G,X) \\ &= \sum_{j=i}^{\dim X} (-1)^j \binom{j+1}{i+1} T_j(X). \end{aligned}$$

Then we have obtain the geometric cone relations for an arbitrary polyhedron of the following theorem.

Theorem 2 *Let X be a polyhedron with a regular decomposition $\mathcal{D}$, then for each $i = 0,1,\cdots,\dim X$,*

$$\bar{T}_i^-(X) = \sum_{j=i}^{\dim X} (-1)^j \binom{j+1}{i+1} T_j(X). \qquad (4)$$

Q.E.D.

Let the closure operator act on (**??**). We then have

Proposition 3

$$\sum_{F \le G \in \mathcal{F}} (-1)^{\dim G} \bar{T}(G, X) = T(-F, -X) = T^{-}(F, X). \tag{5}$$

Q.E.D.

If we take the sum again on both sides of (**??**) over all i-dimensional faces of X, we obtain the geometric relations with closed tangent cones.

Theorem 4 *If X is a polyhedron with a regular decomposition $\mathcal{D}$, then for each $i = 0, 1, \cdots, \dim X$,*

$$T_i^{-}(X) = \sum_{j=i}^{\dim X} (-1)^j \binom{j+1}{i+1} \bar{T}_j(X). \tag{6}$$

Q.E.D.

With the generalized Gauss-Bonnet formula [Ch] there are dual versions of (**??**) and (**??**). These formulas can be obtained by applying the dual operator $*$ to both sides of the (**??**). Note that $(1_C)^* = (-1)^{\dim C} 1_{C^*}$ for any relatively open convex polyhedral cone C. Then

$$\begin{aligned} T^*(F, P) &= (-1)^{\dim P} \bar{K}(F, P) (-1)^{\dim C^*(F,P)} \\ &= (-1)^{\dim V + \dim P - \dim F} \bar{K}(F, P) \\ &= (-1)^{\dim V} \bar{K}'(F, P). \end{aligned}$$

$$\begin{aligned} \bar{T}^*(F, P) &= (-1)^{\dim P} \sum_{F \le G \le P} T^*(F, G) \\ &= (-1)^{\dim V - \dim P} \sum_{F \le G \le P} \bar{K}'(F, G). \\ &= (-1)^{\dim P - \dim F} K(-F, -P) \\ &= K'^{-}(F, P). \end{aligned}$$

Thus we have the following prposition for exterior cones functions.

Proposition 5 *For any polyhedron with a regular decomposition $\mathcal{D}$,*

$$\sum_{F \leq G \in \mathcal{F}} (-1)^{\dim G} \bar{K}'(G, X) = (-1)^{\dim V} K'(F, X). \tag{7}$$

When we apply the closure operator on both sides of (**??**) we have

Proposition 6 *For any polyhedron X with a redular decomposition $\mathcal{D}$,*

$$\sum_{F \leq G \in \mathcal{F}} (-1)^{\dim G} K'(G, X) = (-1)^{\dim V} \bar{K}'(F, X). \tag{8}$$

We thus have the exterior cone function relations.

Theorem 7 *If X is a polyhedron with a simplicial decomposition, then for each $i = 0, 1, \cdots, \dim X$,*

$$\sum_{j=i}^{\dim X} (-1)^j \binom{j+1}{i+1} K_j'(X) = (-1)^{\dim V} \bar{K}_i'(X). \tag{9}$$

$$\sum_{j=i}^{\dim X} (-1)^j \binom{j+1}{i+1} \bar{K}_j'(X) = (-1)^{\dim V} K_i'(X). \tag{10}$$

We consider numerical angles of these tangent cones of X.

Let C be a k-dimensional relatively open convex cone of $\mathbf{V}$. The angle $\alpha(C)$ of C is defined as the normalized k-dimensional Lebesgue measure of C in the unit ball of $\langle C \rangle$, i.e.,

$$\alpha(C) = \mathrm{vol}_k(C \cap B(o, 1))/\mathrm{vol}_k(B(o, 1)).$$

For each relatively open polyheron pair $F \leq P$, we denote the angles of the cones $C(F, P)$ and $C^*(F, P)$ by $\alpha(F, P)$ and $\beta(F, P)$ respectively. Note that the angles $C^\wedge(F, P)$ and $C^\vee(F, P)$ are the same as $\alpha(F, P)$ and $\beta(F, P)$.

For a single relatively open convex polyhedron $P \in \mathcal{D}$ such that $F \leq P$, as a special case of the equation (), we have

$$\sum_{F \leq G \leq P} (-1)^{\dim G} \bar{T}(G, P) = T^-(F, P).$$

Integrate both sides of the equation above, we then have

$$\sum_{F \leq G \leq P} \alpha'(G,P) = \alpha(F,P).$$

For each $F \in \mathcal{F}(X,\mathcal{D})$, if the tangent cone functions of X near F are defined as

$$\alpha(F,X) = \sum_{F \leq P \in \mathcal{D}} \alpha(F,P),$$

$$\alpha'(F,X) = \sum_{F \leq P \in \mathcal{D}} \alpha'(F,P),$$

we thn have the proposition.

Proposition 8

$$\sum_{F \leq G \in \mathcal{F}} \alpha'(G,X) = \alpha(F,X). \tag{11}$$

Let the face F in (**??**) be extended over all i-dimensional faces of X, we then have

Theorem 9 *Let X be a polyhedron with a simplicial decomposition $\mathcal{D}$, then for $i = 0, 1, \cdots, \dim X$,*

$$\alpha_i(X) = \sum_{j=i}^{\dim X} \binom{j+1}{i+1} \alpha'_j(X).$$

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