Eulerian Stratification of Polyhedra

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Contents

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Abstract

In this paper we introduce Eulerian stratifications, weight functions, and boundary weight functions to study linear conditions on f-vectors of stratified triangulations on arbitrary polyhedra. The Eulerian stratified spaces are characterized by the Euler characteristics of the links between strata. With the new concepts, the classical Dehn-Sommerville equations are generalized to weighted f-vectors of arbitrary polyhedra; and the linear conditions on weighted f-vectors of all stratified triangulations are classified. Moreover, the necessary and sufficient conditions are obtained for given numbers to be the Euler characteristics of the links between strata, and a procedure of constructing Eulerian stratifications from the given numbers on posets is provided.

Our study of Eulerian stratifications and weight functions suggest that the underlying combinatorial information should become a rather general setup that includes many classical linear combinatorial theories on f-vectors. It also points out a possible approach towards the study on f-vectors of triangulations of more general spaces.

1 Introduction

Let X be an n-dimensional compact polyhedron. For any triangulation Δ of X, define the f-vector $f(X, \Delta) = (f_0, f_1, \cdots, f_n)$, where f_i is the number of i-simplices of Δ. The f-vectors must satisfy some conditions, such as the Euler equation

$$\chi(f) = f_0 - f_1 + \cdots + (-1)^n f_n = \chi(X).$$

The classification of f-vectors for certain classes of polyhedra and their triangulations is an interesting and difficult problem. The conditions on the f-vectors can be linear, nonlinear, equality, or inequality. For simplicial polytopes, the linear equality conditions have been known (besides the Euler equation) for a long time as the Dehn-Sommerville equations; the linear inequality conditions were formulated by McMullen [13] and were proved to be sufficient by Billera and Lee [2] and to be necessary by Stanley [18]. The classification of f-vectors for arbitrary polyhedra is far from completion.

For topological spaces other than polytopes, Klee [12] first classified linear (equality) conditions on the f-vectors of Eulerian manifolds. Klee defined Eulerian n-manifolds as simplicial complexes such that the simplicial link of every i-simplex has the same Euler characteristic as the sphere S^{n-i-1}. Following Klee's work, the authors [7] [8] showed that the notion of Eulerian manifolds is actually independent of triangulations. Furthermore, we studied the more general two-strata spaces (including manifolds with boundaries) and classified the (rational as well as integral) linear conditions on f-vectors of such spaces.

In this paper we continue to study linear conditions on f-vectors of arbitrary polyhedra. By introducing the notions of Eulerian stratification and weighted f-vector, we are able to obtain the Dehn-Sommerville equations for weighted f-vectors and the (rational) classification of linear conditions on weighted f-vectors. Moreover, we also find the necessary and sufficient conditions for some combinatorial data to be realized as coming from an Eulerian stratification. This last result suggests a general framework that might include many linear combinatorial theories other than f-vectors.

1.1 Preliminaries

We will make use of many concepts (polyhedron, link, join, manifold, triangulation, etc.) from PL-topology. Our basic reference for this is Rourke and Sanderson's classical introduction [16].

A neighborhood of a point x inside a compact polyhedron X is homeomorphic to a cone xL with cone point x and base L. L is itself a compact polyhedron unique up to PL-homeomorphism. L is called the *link* of x in X and is denoted $\mathrm{lk}(x, X)$. The following notions are defined in terms of the link and the Euler characteristic χ:

1. A locally compact polyhedron M is called an n-dimensional PL-*manifold* with a closed subpolyhedron ∂M as boundary if $\mathrm{lk}(x, M)$ is PL-homeomorphic to

the sphere S^{n-1} for $x \in M - \partial M$ and is PL-homeomorphic to the disk D^{n-1} for $x \in \partial M$;

2. A locally compact polyhedron M is called an n-dimensional *Eulerian manifold* with a closed subpolyhedron ∂M as boundary if $\chi(\mathrm{lk}(x, M)) = \chi(S^{n-1}) = 1 - (-1)^n$ for $x \in M - \partial M$ and $\chi(\mathrm{lk}(x, M)) = \chi(D^{n-1}) = 1$ for $x \in \partial M$.

Klee's Eulerian manifolds [12] correspond to our Eulerian manifolds without boundary. We have shown in [8] that as far as Euler characteristic is concerned, Eulerian manifolds with boundary have the same properties as PL-manifolds with boundary. In particular, we proved that for a compact Eulerian manifold $(M, \partial M)$,

$$\chi(\partial M) = (1 - (-1)^{\dim M})\chi(M). \tag{1}$$

We also proved that $\chi(M) = 0$ for any odd dimensional Eulerian manifold without boundary.

In this paper we need to deal with spaces that are not polyhedron in the sense of [16]. They will always be explicitly written as $X = U - V$ for some pair (U, V) of compact polyhedra. We will call X *finite (open) polyhedron*. Many usual notions for compact polyhedra can be extended to these spaces. For example, a *triangulation* of X is the restriction of a triangulation of the pair (U, V), so that X is the union of the interior of some simplices. The *Euler characteristic* $\chi(X) = \chi(U) - \chi(V)$ is still the alternating sum of the number of these simplices at various dimensions. The usual additivity and multiplicativity of the Euler characteristic are still true. Moreover, if $(M, \partial M)$ is a compact Eulerian manifold, then $\dot{M} = M - \partial M$ is a finite polyhedron and

$$\chi(\dot{M}) = \chi(M) - \chi(\partial M) = (-1)^{\dim M}\chi(M). \tag{2}$$

A notable special case is that the Euler characteristic of the n-dimensional open ball is $(-1)^n$.

The *link* $\mathrm{lk}(x, X) = \mathrm{lk}(x, U) - \mathrm{lk}(x, V)$ of $x \in U$ in X is also a finite polyhedron. Here we make use of the notion of the link pair (which is unique up to relative PL-homeomorphism) $(\mathrm{lk}(x, U), \mathrm{lk}(x, V))$ of compact polyhedra. Note that x is not necessarily in X. In fact, $\mathrm{lk}(x, X)$ is nonempty as long as x is in the closure of X in U.

Remark 1.1.1 It is possible to develop a rigorous theory (along the line of [16]) about the spaces that are unions of the interior of some simplices in a triangulation. However, since all spaces encountered in this paper are of the form $U - V$ (and U and V are always explicitly given), we will not get into this.

The classical Dehn-Sommerville equations

$$(1 - (-1)^{n-i})f_i(X, \Delta) + \sum_{j>i}(-1)^{n-j-1}\binom{j+1}{i+1} f_j(X, \Delta) = 0, \quad 0 \le i < n \tag{3}$$

are satisfied by any triangulation of an n-dimensional compact Eulerian manifold X without boundary (see [3] [9] for simplicial polytopes, and [7] [8] [12] for Eulerian manifolds without boundary). Let $D(n)$ denote the coefficient matrix in (3) (see [7] for explicit expression). In [7] [8], we have generalized the classical Dehn-Sommerville equations to $D(n)f(M,\Delta) = f(\partial M, \partial\Delta)$ for compact Eulerian manifolds with boundary. Moreover, we proved the following properties of $D(n)$:

$$D(n-1)D(n) = 0, \quad \chi D(n) = (1-(-1)^n)\chi. \tag{4}$$

Here χ is the Euler function on vectors: $\chi(a_0, a_1, \cdots, a_n) = \sum(-1)^i a_i$.

1.2 Eulerian Stratification

We first introduce the notion of stratified polyhedra.

Definition 1.2.1 *A* stratification *of a compact polyhedron X indexed by a finite partially ordered set P is a decomposition $X = \bigsqcup_{a\in P} X_a$ into disjoint union of finite subpolyhedra such that for each $a \in P$, the closure*

$$\bar{X}_a = \bigsqcup_{b\le a} X_b. \tag{5}$$

We also call X a stratified polyhedron*. Moreover, we call X_a and $\bar{X}_a$* (pure) strata *and* closed strata*.*

In terms of the closed strata $\bar{X}_a$, the condition (5) can be rephrased as

$$\bar{X}_a \cap \bar{X}_b = \bigcup_{c\le a, c\le b} \bar{X}_c.$$

In particular, we see that $\bar{X}_a \subset \bar{X}_b$ if and only if $a \le b$. The strata may be recovered from

$$X_a = \bar{X}_a - \bigcup_{b<a} \bar{X}_b. \tag{6}$$

This gives a specific expression of X_a as a finite polyhedron.

Denote $\chi(a) = \chi(X_a)$ and $\bar{\chi}(a) = \chi(\bar{X}_a)$. Then (5) implies that χ and $\bar{\chi}$ are related by Möbius inversion

$$\bar{\chi}(a) = \sum_{b\le a} \chi(b), \quad \chi(a) = \sum_{b\le a} \bar{\chi}(b)\mu(b,a), \tag{7}$$

where μ is the Möbius function, defined on the ordered pairs of P and characterized by

$$\sum_{a\le c\le b} \mu(c,b) = \begin{cases} 1 & \text{for } a = b \\ 0 & \text{for } a < b. \end{cases}$$

Moreover, we have

$$\chi(X) = \sum_{a\in P} \chi(a).$$

For each $x \in X_a$, the link $\mathrm{lk}(x,X)$ is a stratified polyhedron with strata $\mathrm{lk}(x,X_b)$, indexed by b such that $a \le b$. Moreover, for fixed $a \le b$, $\mathrm{lk}(x,\bar{X}_b)$ is a stratified polyhedron with strata $\mathrm{lk}(x,X_c)$, indexed by c such that $a \le c \le b$.

Definition 1.2.2 *A stratified polyhedron X is called* Eulerian *if*

$$\chi(a,b) = \chi(\mathrm{lk}(x, X_b))$$

is independent of the choice of $x \in X_a$. $\chi(a,b)$ is called the relative Euler characteristic *of X_a in X_b.*

Observe that the notion of Eulerian stratification is purely topological. In fact, for fixed $x \in X_a$, the system $\{\bar{\chi}(a,b) = \chi(\mathrm{lk}(x, \bar{X}_b)) : a \le b\}$ and the system $\{\chi(\mathrm{lk}(x, X_b)) : a \le b\}$ are also related by Möbius inversion in a way similar to (7)

$$\bar{\chi}(a,b) = \sum_{a \le c \le b} \chi(a,c), \qquad \chi(a,b) = \sum_{a \le c \le b} \bar{\chi}(a,c)\mu(c,b), \tag{8}$$

Therefore, one system is independent of the choice of $x \in X_a$ if and only if the other system is independent of the choice. Since the first system can be described in terms of the relative homology

$$\chi(\mathrm{lk}(x, \bar{X}_b)) = 1 - \sum (-1)^i \dim H_i(\bar{X}_b, \bar{X}_b - x),$$

we see Eulerian stratification is a topological property.

Denote $d(a) = \dim X_a$. For a point x in the interior of a $d(a)$-dimensional simplex of a triangulation of X_a, we have $\mathrm{lk}(x, X_a) = S^{d(a)-1}$. Thus from the definition of Eulerian stratified polyhedron we see that for any $y \in X_a$,

$$\chi(\mathrm{lk}(y, X_a)) = \chi(a,a) = \chi(\mathrm{lk}(x, X_a)) = \chi(S^{d(a)-1}) = 1 - (-1)^{d(a)}. \tag{9}$$

This simply means that X_a is an Eulerian manifold without boundary. Hence Eulerian stratified polyhedra are obtained by gluing pieces of Eulerian manifolds together in "Eulerian fashion".

Topologists have been interested in various versions of stratifications such as geometric stratification [4], Whitney stratification [10], PL-stratification [22], intrinsic stratification [14], and homotopy stratification [15]. All these stratifications share the following homogeneity property: for any $x, y \in X_a$ and $a \le b$, there are homeomorphic neighborhoods of x and y in $\bar{X}_b$. Thus geometrically stratified polyhedra are Eulerian stratified. In particular, with the intrinsic stratifications, the spaces such as polyhedra, algebraic varieties, semialgebraic sets, and subanalytic sets are all Eulerian stratified if they are compact.

In [8], we studied Eulerian stratified spaces with 2-strata. We showed that such spaces (X, X_0) are characterized by the following properties

1. $X - X_0$ and X_0 are Eulerian manifolds without boundary;
2. A neighborhood of X_0 in X is the mapping cylinder of a PL-map $f : E \to X_0$ such that $\chi(f^{-1}(x))$ is independent of the choice of $x \in X_0$.

The properties are comparable to the ones that characterize geometrically stratified spaces with 2-strata. We expect general Eulerian stratified spaces can be characterized in a similar way.

1.3 Main Results

There are three main results in this paper: the Dehn-Sommerville equations, the classification of linear conditions on weighted f-vectors, and the characterization of relative Euler characteristics in Eulerian stratifications.

We call a triangulation Δ of a stratified polyhedron X a *stratified triangulation* if each stratum X_a is a union of the interior of some simplices in Δ. We denote the collection of these simplices by Δ_a. Then we have the *a-th f-vector* $f(X_a, \Delta_a)$ given by

$$f_i(X_a, \Delta_a) = \text{number of } i\text{-dimensional simplices in } \Delta_a,$$

and the a-th Euler characteristic

$$\chi(f(X_a, \Delta_a)) = \chi(a).$$

In terms of the closed strata, Δ induces a triangulation $\bar{\Delta}_a$ of $\bar{X}_a$. This gives rise to the f-vector $f(\bar{X}_a, \bar{\Delta}_a)$, and $\chi(f(\bar{X}_a, \bar{\Delta}_a)) = \bar{\chi}(a)$. The two systems $\{f(X_a, \Delta_a)\}$ and $\{f(\bar{X}_a, \bar{\Delta}_a)\}$ are related by Möbius inversion and are therefore equivalent.

We investigate the system $\{f(X_a, \Delta_a)\}$ through the notion of weighted f-vector.

Definition 1.3.1 *Let X be a stratified polyhedron indexed by P. A* weight *is a function ω on P. For a stratified triangulation Δ of X, the* ω-weighted f-vector *is*

$$f(X, \Delta, \omega) = \sum f(X_a, \Delta_a)\omega(a).$$

The ω-weighted Euler characteristic *is*

$$\chi(X, \omega) = \chi(f(X, \Delta, \omega)) = \sum \chi(a)\omega(a).$$

The dimension *of the weight is*

$$d(\omega) = \max_{\omega(a)\neq 0} d(a).$$

For any integer n, the n-th boundary weight $\partial_n\omega$ *of ω is the weight*

$$\partial_n\omega(a) = [1 - (-1)^n]\omega(a) + (-1)^n \sum_{a\leq b} \chi(a, b)\omega(b). \tag{10}$$

Remark 1.3.2 There is no *a priori* restriction on what values the weight may take. In this paper, rational or real values are needed for Propositions 2.2.4, 2.2.5, and most substantially for Theorem 1.3.4.

Alternatively, we can view ω as a function on X, defined by $\omega(x) = \omega(a)$ for $x \in X_a$. We can also view ω as a function on any stratified triangulation Δ of X, defined by $\omega(\sigma) = \omega(a)$ for $\sigma \in \Delta_a$. Then

$$\begin{aligned} f_i(X, \Delta, \omega) &= \sum_{\sigma\in\Delta, \dim\sigma = i} \omega(\sigma), \\ \chi(X, \omega) &= \sum_{\sigma\in\Delta} (-1)^{\dim\sigma}\omega(\sigma), \\ d(\omega) &= \max_{\omega(\sigma)\neq 0} \dim\sigma. \end{aligned}$$

A special case is to take the constant function $\epsilon(a) = 1$, which yields the classical f-vector $f(X, \Delta) = f(X, \Delta, \epsilon)$. Moreover, from the weight function

$$\epsilon_a(b) = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{if } b \neq a, \end{cases}$$

we recover the a-th f-vector $f(X_a, \Delta_a) = f(X, \Delta, \epsilon_a)$.

The following theorem is a generalization of the classical Dehn-Sommerville equations to weighted f-vectors. In the 2-strata case, we have already obtained such generalization in [7] [8] (in an equivalent formulation, without referring to weight functions).

Theorem 1.3.3 *Let ω be a weight on an n-dimensional Eulerian stratified polyhedron X. Then for any stratified triangulation Δ we have*

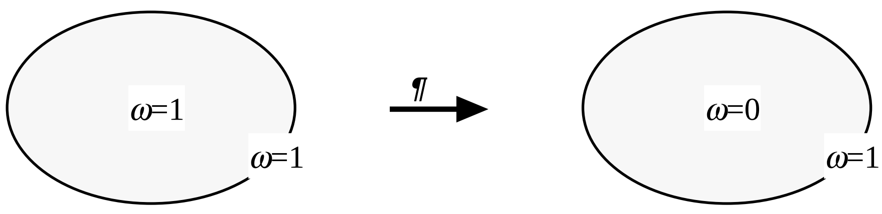
$$D(n)f(X, \Delta, \omega) = f(X, \Delta, \partial_n \omega). \tag{11}$$

One important consequence of the theorem is $\partial_{n-1}\partial_n = 0$ (Proposition 2.2.2). This gives rise to an interesting homology which we will not discuss in this paper. Moreover, this provides one of the conditions in the third main Theorem 1.3.7.

Suppose $n = \dim X = \dim X_a$, then X_a is a top dimensional stratum and a is a maximal index. By (10), therefore, we have

$$\partial_n \omega(a) = [1 - (-1)^{n+d(a)}]\omega(a) = [1 - (-1)^{2n}]\omega(a) = 0.$$

In particular, if X has only one stratum, then X has to be an Eulerian manifold without boundary. Thus (11) becomes $D(n)f(X, \Delta) = 0$, which is the classical Dehn-Sommerville equation.



boundary weight for PL-manifold with boundary

For another example, consider a compact PL-manifold $(M^n, \partial M)$ as a space with strata

$$M_0 = \partial M, \quad M_1 = M - \partial M.$$

We have

$$\begin{aligned} \chi(0,0) &= \chi(S^{n-2}) = 1 + (-1)^n, \\ \chi(0,1) &= \chi(D^{n-1}) = 1, \\ \chi(1,1) &= \chi(S^{n-1}) = 1 - (-1)^n. \end{aligned}$$

Therefore for constant weight function $\epsilon = 1$, we have $\partial_n\epsilon(0) = 1$ and $\partial_n\epsilon(1) = 0$. Then (11) becomes the Dehn-Sommerville equation $D(n)f(M,\Delta) = f(\partial M, \partial\Delta)$ obtained in [7].

The linear conditions on weighted f-vectors are the Euler equation and a part of the Dehn-Sommerville equations. More precisely, for any nonnegative integers $m < n$ of the same parity, there are $(n-m) \times (n-m)$ matrix $E(m,n)$ and $m \times (n-m)$ matrix $F(m,n)$ such that $D(n)$ can be partitioned as follows

$$D(n) = \begin{pmatrix} D(m) & F(m,n) \\ 0 & E(m,n) \end{pmatrix}. \tag{12}$$

Theorem 1.3.4 *Let X be an Eulerian stratified polyhedron.*

1. *For any fixed weight function ω, the only (rational) linear conditions on $v = f(X, \Delta, \omega)$ for all possible stratified triangulations Δ are*

$$\begin{cases} \chi(v) & = \chi(X,\omega) \\ (0, E(r+1,s))v & = 0 \end{cases} \tag{13}$$

   where $s = d(\omega)$ and $r = d(\partial_s\omega)$. Moreover, the number of independent linear conditions is $\frac{s-r+1}{2}$.

2. *Suppose there are some strata whose dimensions are of different parity from* $\dim X$*, and there are some strata with nonvanishing Euler characteristic. Then the only (rational) linear conditions on $v = f(X,\Delta,\omega)$ for all possible stratified triangulations Δ and all weights ω are*

$$(0, E(m+1,n))v = 0$$

   where $n = \dim X$ and m is the greatest dimension of the strata whose parity is different from n. Moreover, the number of independent linear conditions is $\frac{n-m-1}{2}$.

Remark 1.3.5 In (13), v is considered as a vector in $\mathbf{R}^{s+1}$. If v is considered as $(n+1)$-dimensional, then we need to add the extra conditions that the last $n-s$ coordinates of v vanish. Moreover, if $\partial_s\omega \neq 0$, then Proposition 2.2.5 shows that r and s have different parity, so that the statement is meaningful. If $\partial_s\omega = 0$, then we should set $r = 0$ for odd s and $r = -1$ for even s.

Remark 1.3.6 In case the conditions in the second part of the theorem is not satisfied, we have also found all the linear conditions. The details appear in Theorem 3.2.1.

Again these generalize the classical result on the linear conditions on f-vectors. In [8], equivalent results were obtained (without using the notion of weighted f-vectors) for 2-strata case.

Our third main result deals with an important new phenomenon appearing only in stratified situation. It turns out that the Euler characteristic $\chi(a)$ of strata and the relative Euler characteristic $\chi(a, b)$ between strata cannot be arbitrarily given. In addition to (9), the following equalities need to be satisfied (Proposition 2.2.1 and Proposition 2.2.3):

$$\begin{aligned}\sum_{a\leq b}\chi(a)\chi(a,b) &= 0, &&\text{for any fixed } b, &(14)\\ \sum_{a\leq c\leq b}\chi(a,c)\chi(c,b) &= 2\chi(a,b), &&\text{for any fixed } a\leq b. &(15)\end{aligned}$$

Next theorem says that this is also sufficient, modular some lower dimensional quirkyness.

Theorem 1.3.7 *Suppose P is a partially ordered set, and $d : P \to \mathbf{N}$ is a function such that $d(b) - 2 \geq d(a) \geq 1$ for $a < b$. Suppose $\chi(a)$ is a collection of integers for $a \in P$, and $\chi(a, b)$ is another collection of integers for $a \leq b$ in P. Then there exists an Eulerian stratified polyhedron X indexed by P with dimension function $d(a)$, Euler characteristic function $\chi(a)$, and relative Euler characteristic function $\chi(a, b)$ if and only if* (14) *and* (15) *are satisfied, and $\chi(a, a) = 1 - (-1)^{d(a)}$.*

Remark 1.3.8 The reason for requiring the dimension gap is that 0-dimensional polyhedra can only have positive Euler characteristic. In higher dimensions, there is no problem realizing desired Euler characteristics for our specific case.

The theorem is significant because it points out the possible general setup that might include many linear combinatorial theories. Combinatorists have studied Eulerian structures such as Eulerian posets (see [1] [19] [20], for examples). The conditions were imposed on the poset itself. The realization theorem above suggests that we should consider the structures in the more general context of any finite poset with the poset elements related by special functions and incidence functions satisfying (14) and (15).

The other problem that arises from considering stratified polyhedra is that the equality $\partial_{n-1}\partial_n = 0$ gives rise to some "weight homology". It would be quite interesting to find out properties of this homology. We expect it to play some role in the theory speculated above.

In this paper we have not touched the issue of torsion linear conditions. If we take the values of weights to be integers, then the weighted f-vectors are all integral. Therefore there could be some torsion linear relations in addition to the rational linear conditions discussed here (we used rational rank for finding out all the linear conditions, which does not detect torsion relations). The 2-strata case has been completely solved in [8] and was shown to have nontrivial torsion relations. We expect the weight homology to be also involved in this problem.

Lastly, it is of interest to find out whether Theorem 1.3.7 is true for geometrically stratified spaces. A key reason for the theorem to hold is that the cobordism of Eulerian manifolds is extremely simple (see Proposition 4.1.4): $\Omega_n^E = 0$ for odd n and $\Omega_n^E = \mathbf{Z}_2$ for even n. It is quite conceivable that the theorem should also be true for geometrically stratified spaces. But the reason is not obvious.

The rest of the paper consists of detailed proves and is organized as follows: In Section 2 we prove Theorem 1.3.3 on generalized Dehn-Sommerville equations. In Section 3 we study linear conditions on weighted f-vectors and prove Theorem 1.3.4. The lengthy Section 4 is devoted to the realization Theorem 1.3.7.

1.4 Notations

For the convenience of readers, we gather here the various notations used in this paper. For an Eulerian stratified polyhedron X,

$\chi(a) = \chi(X_a)$: Euler characteristic of the stratum X_a
$\chi(a, b)$: relative Euler characteristic of X_a in X_b
$\chi(X, \omega) = \sum \chi(a)\omega(a)$: ω-weighted Euler characteristic of X
$d(a) = \dim X_a$: dimension of the stratum X_a
$d(\omega) = \max\{d(a) : \omega(a) \neq 0\}$: dimension of the weight ω
$D(n)$: Dehn-Sommerville matrix (see [7])
$E(m, n)$: $(n - m) \times (n - m)$ matrix in (12)
$(0, E(m, n))$: matrix with $(m + 1)$ zero columns before $E(m, n)$

The interior of any object A with boundary is denoted $\dot{A}$. Thus the interior of an Eulerian manifold (or PL-manifold) M with boundary ∂M is $\dot{M} = M - \partial M$. The interior of a simplex σ is denoted $\dot{\sigma}$.

The geometric realization of a simplicial complex K is denoted $|K|$. The join of two spaces U and V is

$$U * V = \frac{U \times V \times [0, 1]}{U \times V \times 0 \sim U, U \times V \times 1 \sim V} \cong \text{cone}U \times V \cup_{U \times V} U \times \text{cone}V.$$

Finally, we often find it convenient not to fix the dimension of vectors. We consider vectors in $\mathbf{R}^k$ naturally as vectors in $\mathbf{R}^{k+l}$ by adding l zeros as the $(k + 1)$st through $(k + l)$-th coordinates.

2 Dehn-Sommerville Equations

In this part we derive the generalized Dehn-Sommerville equations for weighted f-vectors stated in Theorem 1.3.3 and discuss its consequences. The method is an elaboration of Klee's argument [12].

2.1 Proof of Dehn-Sommerville Equations

First we recall some facts from PL-topology. Suppose Δ is a triangulation of a compact polyhedron X, and σ is a simplex in the triangulation. Then the *simplicial link* of σ in Δ is

$$\text{lk}(\sigma, \Delta) = \{\tau \in \Delta : \sigma \cap \tau = \emptyset, \tau \text{ and } \sigma \text{ form a simplex } \sigma * \tau \text{ of } \Delta\}. \qquad (16)$$

Moreover, for any $x \in \dot{\sigma}$, we have the following PL-homeomorphism relating the simplicial link and the topological link

$$\mathrm{lk}(x, X) \cong \partial\sigma * |\mathrm{lk}(\sigma, \Delta)|. \tag{17}$$

Now consider a stratified triangulation Δ of a stratified polyhedron X. For $\sigma \in \Delta_a$, the simplicial link $\mathrm{lk}(\sigma, \Delta)$ is also "stratified" with strata

$$\mathrm{lk}(\sigma, \Delta_b) = \{\tau \in \Delta : \sigma \cap \tau = \emptyset, \tau \text{ and } \sigma \text{ form a simplex } \sigma * \tau \in \Delta_b\}, \tag{18}$$

and closed strata

$$\mathrm{lk}(\sigma, \bar{\Delta}_b) = \{\tau \in \Delta : \sigma \cap \tau = \emptyset, \tau \text{ and } \sigma \text{ form a simplex } \sigma * \tau \in \bar{\Delta}_b\}$$

indexed by b such that $a \leq b$. A key step in proving Theorem 1.3.3 is the following computation of the Euler characteristic $\chi(\mathrm{lk}(\sigma, \Delta_b))$.

Proposition 2.1.1 *Let δ be the δ-function on $P \times P$. Then for $\sigma \in \Delta_a$, $a \leq b$, we have*

$$\chi(\mathrm{lk}(\sigma, \Delta_b)) = (1 - (-1)^{\dim\sigma})\delta(a, b) + (-1)^{\dim\sigma}\chi(a, b). \tag{19}$$

Proof: Since $\bar{\Delta}_b$ is a triangulation of compact polyhedron $\bar{X}_b$, and σ is a simplex in $\bar{\Delta}_b$, the relation (17) applies to $x \in \dot{\sigma}$:

$$\mathrm{lk}(x, \bar{X}_b) \cong \partial\sigma * |\mathrm{lk}(\sigma, \bar{\Delta}_b)|.$$

Therefore

$$\begin{aligned} \chi(\mathrm{lk}(x, \bar{X}_b)) &= \chi(\partial\sigma) + \chi(\mathrm{lk}(\sigma, \bar{\Delta}_b)) - \chi(\partial\sigma)\chi(\mathrm{lk}(\sigma, \bar{\Delta}_b)) \\ &= (1 - (-1)^{\dim\sigma}) + \chi(\mathrm{lk}(\sigma, \bar{\Delta}_b)) - (1 - (-1)^{\dim\sigma})\chi(\mathrm{lk}(\sigma, \bar{\Delta}_b)) \\ &= 1 - (-1)^{\dim\sigma} + (-1)^{\dim\sigma}\chi(\mathrm{lk}(\sigma, \bar{\Delta}_b)), \end{aligned}$$

so that

$$\begin{aligned} \chi(\mathrm{lk}(\sigma, \bar{\Delta}_b)) &= 1 - (-1)^{\dim\sigma} + (-1)^{\dim\sigma}\bar{\chi}(a, b) \\ &\overset{(8)}{=} \sum_{a \leq c \leq b} [(1 - (-1)^{\dim\sigma})\delta(a, c) + (-1)^{\dim\sigma}\chi(a, c)]. \end{aligned} \tag{20}$$

On the other hand, from $\bar{\Delta}_b = \bigsqcup_{c \leq b} \Delta_c$ we have $\mathrm{lk}(\sigma, \bar{\Delta}_b) = \bigsqcup_{a \leq c \leq b} \mathrm{lk}(\sigma, \Delta_c)$, which gives rise to

$$\chi(\mathrm{lk}(\sigma, \bar{\Delta}_b)) = \sum_{a \leq c \leq b} \chi(\mathrm{lk}(\sigma, \Delta_c)). \tag{21}$$

By comparing (20) and (21), we obtain (19).

□

Proof of Theorem 1.3.3: Fix $\sigma \in \Delta_a$ and denote $\rho = \sigma * \tau$. Then we have

$$\dim\rho = \dim\sigma + \dim\tau + 1, \tag{22}$$

and the following reinterpretation of (16) and (18)

$$\begin{aligned}\mathrm{lk}(\sigma,\Delta) &\cong \{\rho\in\Delta : \sigma\subset\rho\neq\sigma\},\\ \mathrm{lk}(\sigma,\Delta_b) &\cong \{\rho\in\Delta_b : \sigma\subset\rho\neq\sigma\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\sigma\subset\rho\neq\sigma}(-1)^{\dim\rho+1}\omega(\rho) &= \sum_{a\leq b}\omega(b)\sum_{\rho\in\mathrm{lk}(\sigma,\Delta_b)}(-1)^{\dim\rho+1}\\ &\overset{(22)}{=} \sum_{a\leq b}\omega(b)(-1)^{\dim\sigma}\sum_{\tau\in\mathrm{lk}(\sigma,\Delta_b)}(-1)^{\dim\tau}\\ &= \sum_{a\leq b}\omega(b)(-1)^{\dim\sigma}\chi(\mathrm{lk}(\sigma,\Delta_b))\\ &\overset{(19)}{=} \sum_{a\leq b}[((-1)^{\dim\sigma}-1)\delta(a,b)+\chi(a,b)]\omega(b)\\ &= ((-1)^{\dim\sigma}-1)\omega(a)+\sum_{a\leq b}\chi(a,b)\omega(b). \qquad (23)\end{aligned}$$

If we take the sum of the left side of (23) over all simplices σ of dimension i, then we obtain

$$\begin{aligned}\sum_{\substack{\dim\sigma=i\\ \sigma\subset\rho\neq\sigma}}(-1)^{\dim\rho+1}\omega(\rho) &= \sum_{\dim\rho>i}\sum_{\substack{\sigma\subset\rho\\ \dim\sigma=i}}(-1)^{\dim\rho+1}\omega(\rho)\\ &= \sum_{\dim\rho>i}\binom{\dim\rho+1}{i+1}(-1)^{\dim\rho+1}\omega(\rho)\\ &= \sum_{j>i}(-1)^{j+1}\binom{j+1}{i+1}f_j(X,\Delta,\omega).\end{aligned}$$

Consequently,

$$\sum_{j>i}(-1)^{j+1}\binom{j+1}{i+1}f_j(X,\Delta,\omega)=\sum_{a\in P}\sum_{\substack{\dim\sigma=i\\ \sigma\in\Delta_a}}[((-1)^i-1)\omega(a)+\sum_{a\leq b}\chi(a,b)\omega(b)],$$

and

$$\begin{aligned}&(1-(-1)^{n-i})f_i(X,\Delta,\omega)+\sum_{j>i}(-1)^{n-j-1}\binom{j+1}{i+1}f_j(X,\Delta,\omega)\\ =\ &\sum_{a\in P}\sum_{\substack{\dim\sigma=i\\ \sigma\in\Delta_a}}[(1-(-1)^{n-i})\omega(a)+(-1)^n((-1)^i-1)\omega(a)+(-1)^n\sum_{a\leq b}\chi(a,b)\omega(b)]\\ =\ &\sum_{a\in P}\sum_{\substack{\dim\sigma=i\\ \sigma\in\Delta_a}}[(1-(-1)^n)\omega(a)+(-1)^n\sum_{a\leq b}\chi(a,b)\omega(b)]\\ =\ &\sum_{a\in P}f_i(X_a,\Delta_a)\partial_n\omega(a).\end{aligned}$$

This completes the proof of Theorem 1.3.3.

□

2.2 Consequences of Dehn-Sommerville Equations

We collect here some consequences of Dehn-Sommerville equations derived in the last section.

Proposition 2.2.1 *The Euler characteristic and the relative Euler characteristic of an Eulerian stratified polyhedron satisfies*

$$\sum_{a\le b}\chi(a)\chi(a,b)=0,\quad \textit{for any fixed } b.$$

Proof: By applying the second equality in (4) to the Dehn-Sommerville equations, we obtain

$$\begin{aligned}(1-(-1)^n)\sum\chi(a)\omega(a) &= (1-(-1)^n)\chi(f(X,\Delta,\omega))\\ &= \chi(D(n)f(X,\Delta,\omega))\\ &= \chi(f(X,\Delta,\partial_n\omega))\\ &= \sum(1-(-1)^n)\chi(a)\omega(a)+(-1)^n\sum_{a\le b}\chi(a)\chi(a,b)\omega(b).\end{aligned}$$

Since this is true for any ω, the proposition is proved.

□

Proposition 2.2.2 $\partial_{n-1}\partial_n\omega=0$ *for any weight* ω *on an Eulerian stratified space* X.

Proof: Denote $\theta=\partial_{n-1}\partial_n\omega$. By applying Dehn-Sommerville equations twice and the first equality in (4), we obtain

$$0=D(n-1)D(n)f(X,\Delta,\omega)=f(X,\Delta,\partial_{n-1}\partial_n\omega)=f(X,\Delta,\theta).$$

for any triangulation Δ. Then for any refinement Δ' of Δ we have

$$\sum(f_i(X_a,\Delta'_a)-f_i(X_a,\Delta_a))\theta(a)=f_i(X,\Delta',\theta)-f_i(X,\Delta,\theta)=0.$$

Now let b be a maximal index such that $\theta(b)\neq 0$. Then we may choose a subdivision Δ' of Δ such that $\Delta'_b\neq\Delta_b$, and $\Delta'_a=\Delta_a$ for $a\not\geq b$. As a result, $f_i(X_b,\Delta'_b)>f_i(X_b,\Delta_b)$, and $f_i(X_a,\Delta'_a)=f_i(X_a,\Delta_a)$ for $a\not\geq b$. Combined with the maximality assumption on b, we get

$$\sum(f_i(X_a,\Delta'_a)-f_i(X_a,\Delta_a))\theta(a)=(f_i(X_b,\Delta'_b)-f_i(X_b,\Delta_b))\theta(b)\neq 0.$$

The contradiction implies that $\theta=0$.

□

Proposition 2.2.3 *The relative Euler characteristic of an Eulerian stratified polyhedron satisfies*

$$\sum_{a\le c\le b}\chi(a,c)\chi(c,b)=2\chi(a,b).$$

Proof: We explicitly find out the meaning of $\partial_{n-1}\partial_n = 0$:

$$\begin{aligned}
0 &= \partial_{n-1}\partial_n\omega(a) \\
&= (1-(-1)^{n-1})\partial_n\omega(a) + (-1)^{n-1}\sum_{a\le b}\chi(a,b)\partial_n\omega(b) \\
&= (1-(-1)^{n-1})[(1-(-1)^n)\omega(a) + (-1)^n\sum_{a\le b}\chi(a,b)\omega(b)] \\
&\quad +(-1)^{n-1}\sum_{a\le b}\chi(a,b)[(1-(-1)^n)\omega(b) + (-1)^n\sum_{b\le c}\chi(b,c)\omega(c)] \\
&= \sum_{a\le b} 2\chi(a,b)\omega(b) - \sum_{a\le c\le b}\chi(a,c)\chi(c,b)\omega(b).
\end{aligned}$$

Since this holds for any choice of ω, the proposition is proved.

□

Proposition 2.2.4 *Suppose ω is a nonzero weight function on an Eulerian stratified polyhedron X. If $\partial_n\omega = 0$, then n and $d(\omega)$ have the same parity.*

Proof: Choose a maximal index a such that $d(a) = d(\omega)$ and $\omega(a) \neq 0$. Then by maximality we have $\omega(b) = 0$ for $a < b$. Therefore

$$\begin{aligned}
0 = \partial_n\omega(a) &= (1-(-1)^n)\omega(a) + (-1)^n\chi(a,a)\omega(a) \\
&= (1-(-1)^n)\omega(a) + (-1)^n(1-(-1)^{d(a)})\omega(a) \\
&= (1-(-1)^{n+d(a)})\omega(a).
\end{aligned}$$

Since $\omega(a) \neq 0$ and $d(a) = d(\omega)$, we conclude that $1-(-1)^{n+d(\omega)} = 0$.

□

Proposition 2.2.5 *Given a weight function ω on an Eulerian stratified space X, we have either $\partial_n\omega = 0$, or n and $d(\partial_n\omega)$ have different parity.*

Proof: By Proposition 2.2.2, for any weight ω we may take n to be $n-1$ and ω to be $\partial_n\omega$ in Proposition 2.2.4. The proposition then follows.

□

3 Linear Conditions on Weighted f-vectors

In this part we study the linear conditions on weighted f-vectors. In the first section, we deal with the case of fixed weight. In the second section we consider all weights.

3.1 The Case of Fixed Weight

In this section, we fix a weight ω on an Eulerian stratified polyhedron X and consider the linear conditions on the weighted f-vectors $f(X, \Delta, \omega)$ for all stratified triangulations Δ. Such linear conditions are explicitly given in Theorem 3.1.1, which is the first part of Theorem 1.3.4.

In the proof of Theorem 3.1.1, we will make use of the existence of triangulations δ_i^p of D^p, $0 \leq i \leq \lfloor \frac{p+1}{2} \rfloor$, with the following properties

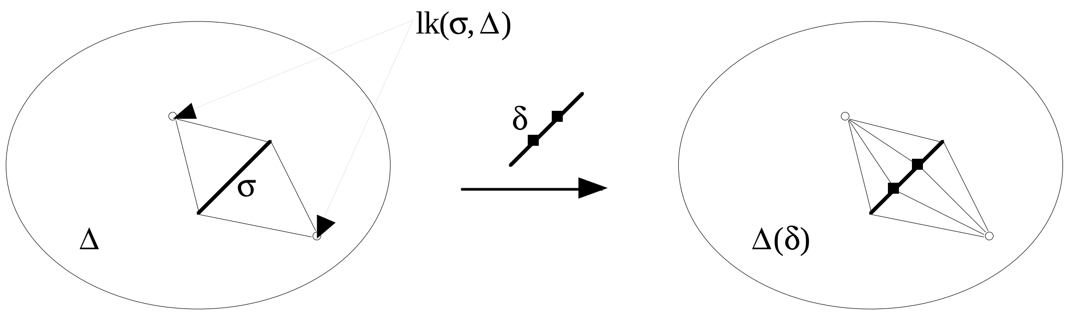
1. $\partial \delta_i^p = \delta_i^p|_{S^{p-1}} = \partial \Delta^p$, the boundary of the standard p-simplex;
2. The f-vectors $f(D^p, \delta_i^p)$ are affinely independent.

Such triangulations may be for instance obtained by deleting one p-dimensional simplex from each of the boundary triangulations of $(p+1)$-dimensional cyclic polytopes [3] [9] [23].

The proof of the theorem is very much similar to the ones found in [7] and [8]. The key construction is the following: Suppose Δ is a triangulation of X and σ is a p-dimensional simplex in the triangulation. Then nearby σ is the subcomplex

$$\sigma * \mathrm{lk}(\sigma, \Delta) = \{\nu * \tau : \nu \text{ is a face of } \sigma, \tau \in \mathrm{lk}(\sigma, \Delta)\}.$$

For each special triangulation $\delta = \delta_i^p$, we may replace the subcomplex $\sigma * \mathrm{lk}(\sigma, \Delta)$ of Δ by the complex $\delta * \mathrm{lk}(\sigma, \Delta)$. Since $\sigma * \mathrm{lk}(\sigma, \Delta)$ is glued to the rest of Δ along $\partial\sigma * \mathrm{lk}(\sigma, \Delta) = \partial\delta * \mathrm{lk}(\sigma, \Delta)$ (the equality is by the first property of δ), after the replacement we still obtain a triangulation of X. We denote the triangulation by $\Delta(\delta)$.



The following repeats the first part of Theorem 1.3.4.

Theorem 3.1.1 *Let ω be a rational weight function on an Eulerian stratified polyhedron X. Let $s = d(\omega)$ and $r = d(\partial_s\omega)$. Then the rational affine span of the ω-weighted f-vectors $f(X, \Delta, \omega)$ for all possible stratified triangulations Δ of X consists of vectors $v \in \mathbf{R}^{s+1}$ satisfying*

$$\begin{cases} \chi(v) & = \chi(X, \omega) \\ (0, E(r+1, s))v & = 0 \end{cases}$$

Moreover, the number of independent linear conditions is $\frac{s-r+1}{2}$, and the dimension of the affine span is $\frac{s+r+1}{2}$.

Proof: Because $d(\omega) = s$, $f(X, \Delta, \omega)$ is really an $(s+1)$-dimensional vector. And we will take such a viewpoint in the subsequent proof.

Let A be the affine span of the f-vectors $f(X, \Delta, \omega)$ for various Δ. Let B be the affine subspace determined by the two linear conditions. We already know that all ω-weighted f-vectors satisfy the Euler equation and the Dehn-Sommerville equations $D(s)f(X, \Delta, \omega) = f(X, \Delta, \partial_s\omega)$. Because $d(\partial_s\omega) = r$, we have $f_i(X, \Delta, \partial_s\omega) = 0$ for all $i > r$. In other words, the vector $(0, E(r+1, s))f(X, \Delta, \omega)$ vanishes. Thus we conclude that $A \subset B$.

We have shown in [8] that the dimension of B is $\frac{s+r+1}{2}$. In fact, this follows from the classical result in [3] [9] on the rank of the system of the Euler equation and the whole Dehn-Sommerville equations. Therefore to show $A = B$, it suffices to find $\frac{s+r+1}{2} + 1$ triangulations so that their ω-weighted f-vectors are affinely independent.

Let a be a maximal index such that $d(a) = r = d(\partial_s\omega)$ and $\partial_s\omega(a) \neq 0$. Let b be a maximal index such that $d(b) = s = d(\omega)$ and $\omega(b) \neq 0$. We fix a stratified triangulation Δ of X. By refining Δ, we may further assume that there are closed simplices σ^r and τ^s contained in strata X_a and X_b.

We make use of the special triangulations δ_i^r of D^r, $0 \le i \le \lfloor \frac{r+1}{2} \rfloor$, to construct new triangulations

$$\Delta_i = \Delta(\delta_i^r), \quad 0 \le i \le \lfloor \tfrac{r+1}{2} \rfloor. \tag{24}$$

Recall that $\Delta(\delta_i^r)$ is obtained by replacing the subcomplex $\sigma * \mathrm{lk}(\sigma, \Delta)$ with the complex $\delta_i^r * \mathrm{lk}(\sigma, \Delta)$. Because all faces of σ^r are assumed to be contained in X_a, and $r = d(\partial_s\omega)$, by dimension reason the only modification on those strata with nonvanishing $\partial_s\omega$ is the replacement of σ by δ_i^r. Therefore by the first property of δ_i^r, the $\partial_s\omega$-weighted f-vectors of the new triangulations are

$$f(X, \Delta_i, \partial_s\omega) = f(X, \Delta, \partial_s\omega) + [f(D^r, \delta_i^r) - f(S^{r-1}, \partial\Delta^r)]\partial_s\omega(a).$$

By the second property of δ_i^r, $f(D^r, \delta_i^r)$ are affinely independent. Thus we conclude from $\partial_s\omega(a) \neq 0$ that the weighted f-vectors

$$f(X, \Delta_i, \partial_s\omega), \quad 0 \le i \le \lfloor \tfrac{r+1}{2} \rfloor \tag{25}$$

are also affinely independent.

Next we do the similar construction to τ. We make use of the special triangulations δ_j^s of D^s, $0 \le j \le \lfloor \frac{s+1}{2} \rfloor$, to create new triangulations

$$\Delta_{0,j} = \Delta_0(\delta_j^s), \quad 0 \le j \le \lfloor \tfrac{s+1}{2} \rfloor \tag{26}$$

from the 0-th triangulation Δ_0 of (24). Because all faces of τ^s are assumed to be contained in X_b, and $s = d(\omega)$, again by dimension reason the only modification on the strata with nonvanishing ω is the replacement of τ by δ_j^s. Thus

$$f(X, \Delta_{0,j}, \omega) = f(X, \Delta_0, \omega) + [f(D^s, \delta_j^s) - f(S^{s-1}, \partial\Delta^s)]\omega(b).$$

By the second property of δ_j^s, $f(D^s, \delta_j^s)$ are affinely independent. Thus we conclude from $\omega(b) \neq 0$ that the weighted f-vectors

$$f(X, \Delta_{0,j}, \omega), \quad 0 \leq j \leq \lfloor \tfrac{n+1}{2} \rfloor \tag{27}$$

are also affinely independent.

Another fact we need about the f-vectors (27) is

$$D(s)[f(X, \Delta_{0,j}, \omega) - f(X, \Delta_0, \omega)] = f(X, \Delta_{0,j}, \partial_s\omega) - f(X, \Delta_0, \partial_s\omega) = 0. \tag{28}$$

Here in the last step, we use the fact that the difference between $\Delta_{0,j}$ and Δ_0 appears near τ, which by dimension reason is assumed to be away from strata with nonvanishing $\partial_s\omega$.

Now we replace the 0-th triangulation in (24) by the sequence of triangulations (26) created from it, and consider the list

$$\Delta_1, \Delta_2, \cdots, \Delta_{\lfloor \frac{r+1}{2} \rfloor}; \quad \Delta_{0,0}, \Delta_{0,1}, \cdots \Delta_{0, \lfloor \frac{s+1}{2} \rfloor}.$$

Since r and s have different parity, the number of triangulations in the list is

$$\left\lfloor \frac{r+1}{2} \right\rfloor + 1 + \left\lfloor \frac{s+1}{2} \right\rfloor = \frac{s+r+1}{2} + 1.$$

We claim that their ω-weighted f-vectors are affinely independent, which implies the theorem.

So we consider the linear relation

$$\sum_{i=1}^{\lfloor \frac{r+1}{2} \rfloor} \alpha_i[f(X, \Delta_i, \omega) - f(X, \Delta_{0,0}, \omega)] + \sum_{j=1}^{\lfloor \frac{s+1}{2} \rfloor} \beta_j[f(X, \Delta_{0,j}, \omega) - f(X, \Delta_{0,0}, \omega)] = 0. \tag{29}$$

We need to show that all the coefficients α_i, β_j vanish.

Applying $D(s)$ to (29), we use (28) and the Dehn-Sommerville equations to obtain

$$\sum_{i=1}^{\lfloor \frac{r+1}{2} \rfloor} \alpha_i[f(X, \Delta_i, \partial_s\omega) - f(X, \Delta_0, \partial_s\omega)] = 0.$$

By the affine independence of (25), we see that all the coefficients $\alpha_i = 0$. Thus the relation (29) becomes

$$\sum_{j=1}^{\lfloor \frac{s+1}{2} \rfloor} \beta_j[f(X, \Delta_{0,j}, \omega) - f(X, \Delta_{0,0}, \omega)] = 0.$$

By the affine independence of (27), we further conclude that all the coefficients $\beta_j = 0$.

This completes the proof of Theorem 3.1.1.

$\square$

3.2 The Case of Variable Weight

In this section, we fix an Eulerian stratified polyhedron X and consider the linear conditions on the weighted f-vectors $f(X, \Delta, \omega)$ for all weights ω and all stratified triangulations Δ. Our main purpose is to prove the following theorem, which includes the second part of Theorem 1.3.4.

Theorem 3.2.1 *Let X be an n-dimensional Eulerian stratified polyhedron. Then depending on various situations, the rational affine span of the f-vectors $f(X, \Delta, \omega)$ for all weights ω and all stratified triangulations Δ of X is characterized by the following equations:*

- *Suppose there exist some strata whose dimensions are of different parity from n. Let m be the greatest dimension of such strata.*

  1. *In case some $\chi(a) \neq 0$,*
  $$(0, E(m+1, n))v = 0, \tag{30}$$
  and the number of independent linear conditions is $\frac{n-m-1}{2}$.
  2. *In case all $\chi(a) = 0$,*
  $$\begin{cases} \chi(v) & = & 0 \\ (0, E(m+1, n))v & = & 0, \end{cases} \tag{31}$$
  and the number of independent linear conditions is $\frac{n-m+1}{2}$.

- *Suppose the dimensions of all strata are of the same parity.*

  1. *In case some $\chi(X_a) \neq 0$,*
  $$D(n)v = 0, \tag{32}$$
  and the number of independent linear conditions is $\lfloor \frac{n+1}{2} \rfloor$.
  2. *In case all $\chi(X_a) = 0$,*
  $$\begin{cases} \chi(v) & = & 0 \\ D(n)v & = & 0, \end{cases} \tag{33}$$
  and the number of independent linear conditions is $\lceil \frac{n+1}{2} \rceil$.

Proof: We first consider the case that there exist some strata whose dimensions are of different parity from n. Obviously, the Euler equation $\chi(v) = 0$ is satisfied by all f-vectors in case all $\chi(a) = 0$. To show that $f(X, \Delta, \omega)$ always satisfies $(0, E(m+1, n))v = 0$, we need to take into account of the parity of $d(\omega)$.

If $d(\omega)$ and n have the same parity, then by Proposition 2.2.5, $d(\partial_n \omega)$ and n must have different parity. Thus $d(\partial \omega) \leq m$ by the definition of m. Consequently, the coordinates of $D(n)f(X, \Delta, \omega) = f(X, \Delta, \partial_n \omega)$ vanish at dimensions $> m$. These coordinates are exactly $(0, E(m+1,n))f(X,\Delta,\omega)$. Therefore the f-vector satisfies $(0, E(m+1,n))v = 0$.

If $d(\omega)$ and n have different parity, then $d(\omega) \le m$ by the definition of m. Consequently, the coordinates of $f(X,\Delta,\omega)$ vanish at dimensions $> m$. Since the 0 in the matrix $(0, E(m+1,n))$ represents the first $m+1$ columns, we conclude that $(0, E(m+1,n))f(X,\Delta,\omega) = 0$.

So we have proved that either (30) or (31) is satisfied by all f-vectors in respective situation. To show that these are all the linear conditions, we only need to find some weights so that the affine span of their f-vectors is exactly given by these equations.

Suppose $\chi(a) \neq 0$ for some a. Then we fix one such stratum X_a. Moreover, we fix an n-dimensional stratum X_b and an m-dimensional stratum X_c. Then we are looking for a weight ω_0 of the form

$$\omega_0(p) = \begin{cases} \alpha & p = a \\ \beta & p = b \\ \gamma & p = c \\ 0 & \text{otherwise,} \end{cases}$$

where we may additionally require $\alpha = \beta$ if $a = b$ or $\alpha = \gamma$ if $a = c$. We can always find appropriate α, β, γ so that the following are satisfied.

$$\chi(X,\omega_0) = \chi(a)\alpha + \chi(b)\beta + \chi(c)\gamma \neq 0, \tag{34}$$

$$\omega_0(b) = \beta \neq 0, \tag{35}$$

$$\partial_n\omega_0(c) = \begin{cases} 2\gamma + (-1)^n(\chi(c,a)\alpha + \chi(c,b)\beta) & \text{if } c < a, c < b \\ 2\gamma + (-1)^n\chi(c,b)\beta & \text{if } c \not< a, c < b \\ 2\gamma + (-1)^n\chi(c,a)\alpha & \text{if } c < a, c \not< b \\ 2\gamma & \text{if } c \not< a, c \not< b \end{cases} \neq 0. \tag{36}$$

Now (35) means $d(\omega_0) = n$, and (36) means $d(\partial_n\omega_0) = m$. Thus by Theorem 3.1.1, the affine span of ω_0-weighted f-vectors $f(X,\Delta,\omega_0)$ for all triangulations is characaterized by $(0, E(m+1,n))v = 0$ and $\chi(v) = \chi(X,\omega_0)$. Similarly, the affine span of $2\omega_0$-weighted f-vectors $f(X,\Delta,\omega_0)$ for all triangulations is characaterized by $(0, E(m+1,n))v = 0$ and $\chi(v) = 2\chi(X,\omega_0)$. Since $\chi(X,\omega_0) \neq 0$ by (34), the affine span of ω_0-weighted and $2\omega_0$-weighted f-vectors for all triangulations is characaterized by (30) only.

Suppose $\chi(a) = 0$ for all a. Then we fix an n-dimensional stratum X_b and an m-dimensional stratum X_c. Then we are looking for a weight ω_0 of the form

$$\omega_0(p) = \begin{cases} \beta & p = b \\ \gamma & p = c \\ 0 & \text{otherwise,} \end{cases}$$

We can always find appropriate β, γ so that the following are satisfied.

$$\omega_0(b) = \beta \neq 0,$$

$$\partial_n\omega_0(c) = \begin{cases} 2\gamma + (-1)^n\chi(c,b)\beta & \text{if } c < b \\ 2\gamma & \text{if } c \not< b \end{cases} \neq 0.$$

As before, these inequalities imply that $d(\omega_0) = n$ and $d(\partial_n \omega_0) = m$. By Theorem 3.1.1, the affine span of ω_0-weighted f-vectors $f(X, \Delta, \omega_0)$ for all triangulations is characaterized by (31).

From Theorem 3.1.1 we know the rank of (31) is $\frac{n-m+1}{2}$. Since $m \geq 0$, the first column of $(0, E(m+1, n))$ is zero. Therefore the Euler equation is linearly independent of the equations $(0, E(m+1, n))v = 0$. As a result, the rank of $(0, E(m+1, n))$ is $\frac{n-m+1}{2} - 1 = \frac{n-m-1}{2}$.

This completes the proof of the first part of the theorem. Now we turn to the case that all strata have the same parity.

For any weight ω, we must have $\partial_n \omega = 0$ by Proposition 2.2.5. Then by the Dehn-Sommerville equations (11), $D(n)f(X, \Delta, \omega) = 0$ for any triangulation Δ. Moreover, the Euler equation $\chi(f(X, \Delta, \omega)) = 0$ is obviously satisfied in case $\chi(a) = 0$ for all a. So we have proved that either (32) or (33) is satisfied by all f-vectors in respective situation. As in the first part, it remains to find some weights so that the affine span of their f-vectors is exactly given by these equations.

Suppose $\chi(a) \neq 0$ for some a. Then we fix one such stratum X_a and an n-dimensional stratum X_b. By choosing values $\omega_0(a)$ and $\omega_0(b)$ carefully, and taking ω_0 to be 0 elsewhere, we can find ω_0 so that $\chi(X, \omega_0) \neq 0$ and $d(\omega) = n$. A similar argument as before shows that the affine span of ω_0-weighted and $2\omega_0$-weighted f-vectors for all triangulations is characaterized by (32) only.

Suppose $\chi(a) = 0$ for all a. Then we can find a weight ω_0 with $d(\omega_0) = n$. By Theorem 3.1.1, the affine span of ω_0-weighted f-vectors $f(X, \Delta, \omega_0)$ for all triangulations is characaterized by (33).

The ranks of (32) and (33) are well known (see [3] [9] or [23], for examples).

□

4 Characterization of Eulerian Stratifications

In this part we prove Theorem 1.3.7. The necessity of the conditions have been shown in (9), Proposition 2.2.1, and Proposition 2.2.3. The proof of the sufficiency consists of two parts. First we construct pieces Y_* of Eulerian manifolds corresponding to the strata and links between strata. Then we put these together to form a stratified polyhedron X, which we have to show to be Eulerian with the right Euler characteristics.

4.1 Construction of Eulerian Pieces

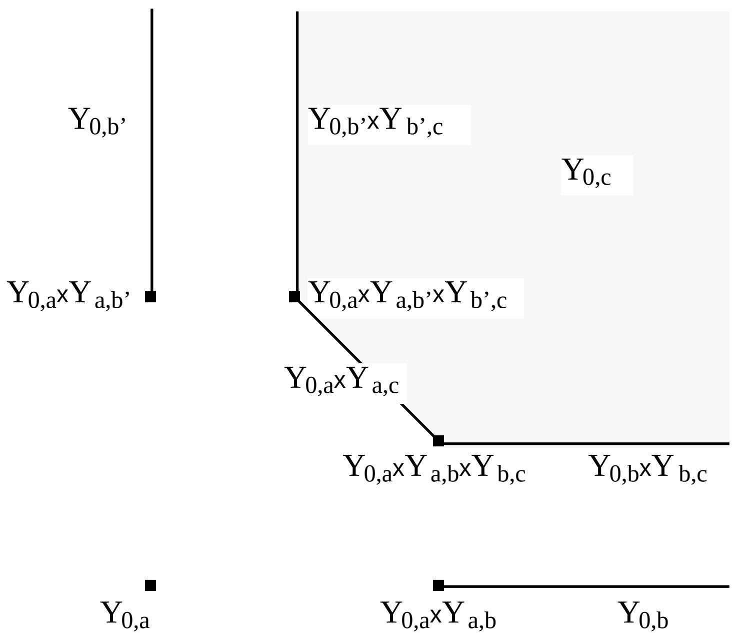
Let $\tilde{P} = P \cup \{0\}$, where 0 is the smallest element joined to P. We may think of 0 as corresponding to the empty stratum $\emptyset$. In this part we construct Eulerian manifolds $Y_{a,b}$ with boundary $\partial Y_{a,b}$ for $a < b$ in $\tilde{P}$. We expect $Y_{0,b}$ to become the "closed interior" of strata of X. For $a < b$ in P, we also expect $Y_{a,b}$ to become (the core of) links between strata. In the next section, we use these pieces to construct X.

To state the properties these Eulerian manifolds $Y_{a,b}$ must satisfy, we need to fix some notations. Throughout the proof, I, J, K, etc, will be *strictly ordered* subsets (called chains in [19]) of P. Moreover,

$$\begin{array}{rcl} |I| & = & \text{number of indices in } I, \\ \Delta^I & = & \text{simplex with vertex set } I, \\ \dot{\Delta}^I & = & \text{interior of the simplex}, \\ i(I) & = & \text{initial index of } I, \\ t(I) & = & \text{terminal index of } I. \end{array}$$

We say J *refines* I if $I \subset J$, and $i(I) = i(J)$, $t(I) = t(J)$. For $I = \{a_0 < a_1 < \cdots < a_r\} \subset \tilde{P}$, denote

$$Y_I = Y_{a_0,a_1} \times Y_{a_1,a_2} \times \cdots \times Y_{a_{r-1},a_r}. \tag{37}$$



Relation between Y I for P={a,b,b',c} with a<b<c, a<b'<c

Lemma 4.1.1 *Suppose $d(b) - 2 \geq d(a) \geq 1$ for $a < b$ in P, and (9), (14), (15) are satisfied. Then there exist Eulerian manifolds $Y_{a,b}$ for $a < b$ in $\tilde{P}$ such that*

1. $Y_I \cap Y_J = \cup_{K \text{ refines } I,J} Y_K$,

2. $\partial Y_I = \cup_{J \text{ refines } I, J \neq I} Y_J$,

3. $\dim Y_{a,b} = d(b) - d(a) - 1$, *where* $d(0) = -1$,

4. $\chi(Y_{0,b}) = (-1)^{d(b)} \chi(b)$ *for* $b \in P$,

5. $\chi(Y_{a,b}) = (-1)^{d(b)-1} \chi(a,b)$ *for* $0 < a < b$.

We note that the expected dimension of Y_I is

$$d(I) = d(t(I)) - d(i(I)) - |I| + 1. \tag{38}$$

We also note that the first two conditions imply that

$$Y_I = \bigsqcup_{J \text{ refines } I} \dot{Y}_J, \tag{39}$$

$$\partial Y_I = \bigsqcup_{J \text{ refines } I, J \neq I} \dot{Y}_J, \tag{40}$$

where $\dot{Y}_J = Y_J - \partial Y_J$ and ∂Y_J is given by the equality in the second condition in the lemma.

Proof of Lemma 4.1.1: For the convenience of applying (14) and (15), we note that if we set

$$\chi(0, a) = \begin{cases} -\chi(a) & \text{if } a \neq 0 \\ 1 - (-1)^{-1} = 2 & \text{if } a = 0. \end{cases} \tag{41}$$

Then (14) is equivalent to (15) with $a = 0$, and the fourth condition becomes the fifth one with $a = 0$. On the other hand, we may use $\chi(a, a) = 1 - (-1)^{d(a)}$ to rephrase the condition into

$$((-1)^{d(a)} + (-1)^{d(b)})\chi(a, b) = \sum_{a<c<b} \chi(a, c)\chi(c, b), \quad \text{for } a < b \text{ in } \tilde{P}. \tag{42}$$

The construction of $Y_{a,b}$ is by induction on the number of indices between a and b. When we say something has been constructed, we assume that the five properties in the lemma are satisfied for the done part.

We start with the construction of $Y_{a,b}$, where $a < b$ are in $\tilde{P}$ and there is no $c \in P$ such that $a < c < b$. The requirement on $Y_{a,b}$ is that it is a $(d(b) - d(a) - 1)$-dimensional Eulerian manifold without boundary and with Euler characteristic $\chi(Y_{a,b}) = (-1)^{d(b)-1}\chi(a, b)$. If $d(b) - d(a) - 1$ is even, then we may always find a PL-manifold with such property, which can be chosen as $Y_{a,b}$. If $d(b) - d(a) - 1$ is odd, then the condition (42) means $(-1)^{d(a)}2\chi(a, b) = 0$. Therefore $\chi(a, b) = 0$ and any closed $(d(b) - d(a) - 1)$-dimensional manifold (such manifolds necessarily have vanishing Euler characteristic) can be chosen as $Y_{a,b}$.

Now fix any pair $a < b$ in $\tilde{P}$. The discussion above allows us to inductively assume that Y_{c_1,c_2} has been constructed for all $a \le c_1 < c_2 \le b$, but $(c_1, c_2) \neq (a, b)$. Consequently, Y_I has been constructed by (37) for any $I \neq \{a, b\}$ with $i(I) = a$ and $t(I) = b$ (i.e., I refines but is not equal to $\{a, b\}$), so that the boundary

$$\partial Y_{a,b} = \bigcup_{I \text{ refines } \{a,b\}, I \neq \{a,b\}} Y_I \tag{43}$$

of $Y_{a,b}$ has been prescribed. We need to show that the prescribed boundary is an Eulerian manifold without boundary. Moreover, we need to show that it has the right Euler characteristic to become the boundary of a $(d(b) - d(a) - 1)$-dimensional Eulerian manifold with Euler characteristic $(-1)^{d(b)-1}\chi(a, b)$. This Eulerian manifold will then be our $Y_{a,b}$.

Proposition 4.1.2 (43) *is a* $(d(b)-d(a)-2)$*-dimensional Eulerian manifold without boundary.*

Proof: Let y be a point in (43). We need to show that $\chi(\mathrm{lk}(y, \partial Y_{a,b})) = \chi(S^{d(b)-d(a)-3})$.

Suppose $y \in \dot{Y}_I$ (see (40)). Then $y \in Y_J$ if and only if $I \supset J \supset \{a,b\}$. Therefore

$$\mathrm{lk}(y, \partial Y_{a,b}) = \bigcup_{I \supset J \supset \{a,b\}, J \neq \{a,b\}} \mathrm{lk}(y, Y_J). \tag{44}$$

Moreover, since $Y_I \subset \partial Y_J$ in case $I \neq J$, and Y_J is inductively assumed to be an Eulerian manifold with boundary ∂Y_J, we have

$$\chi(\mathrm{lk}(y, Y_J)) = \begin{cases} 1 & \text{if } J \neq I \\ 1 - (-1)^{d(I)} & \text{if } J = I, \end{cases}$$

where $d(I) = d(b) - d(a) - |I| + 1$ is given by (38). To compute the Euler characteristic of (44), we compare (44) with the stratified space $S^{d(I)-1} * \partial\Delta^{I-\{a,b\}}$ with closed strata $S^{d(I)-1} * \Delta^K$, $K \subset I - \{a,b\}$. The correspondence

$$\mathrm{lk}(y, Y_J) \Leftrightarrow S^{d(I)-1} * \Delta^K, \quad J \Leftrightarrow K = I - J$$

between the closed strata of (44) and those of $S^{d(I)-1} * \partial\Delta^{I-\{a,b\}}$ preserves the incidence relation

$$\mathrm{lk}(y, Y_{J_1}) \bigcap \mathrm{lk}(y, Y_{J_2}) = \mathrm{lk}(y, Y_{J_1 \cup J_2}),$$

$$(S^{d(I)-1} * \Delta^{I-J_1}) \bigcap (S^{d(I)-1} * \Delta^{I-J_2}) = S^{d(I)-1} * \Delta^{(I-J_1)\cap(I-J_2)} = S^{d(I)-1} * \Delta^{I-J_1 \cup J_2}.$$

Moreover, the corresponding closed strata have the same Euler characteristic

$$\chi(S^{d(I)-1} * \Delta^{I-J}) = \begin{cases} \chi(D^{d(I)-1+|I-J|}) \\ \chi(S^{d(I)-1}) \end{cases} = \begin{cases} 1 & \text{if } J \neq I \\ 1 - (-1)^{d(I)} & \text{if } J = I. \end{cases}$$

By Möbius inversion, the corresponding strata also have the same Euler characteristic. By adding the Euler characteristics of the strata together, we conclude that the Euler characteristics of the total spaces are the same

$$\begin{aligned} \chi(\mathrm{lk}(y, \partial Y_{a,b})) &= \chi(S^{d(I)-1} * \partial\Delta^{I-\{a,b\}}) \\ &= \chi(S^{(d(b)-d(a)-r-1)+1+(r-3)}) \\ &= \chi(S^{d(b)-d(a)-3}). \end{aligned}$$

□

Next we compute the Euler characteristic of (43). By inductive hypothesis, Y_I is an Eulerian manifold. Therefore (2) gives us $\chi(\dot{Y}_I) = (-1)^{d(I)}\chi(Y_I)$. Then by applying (40) to $\{a,b\}$ we get $\partial Y_{a,b} = \bigsqcup_{I \text{ refines } \{a,b\}, I \neq \{a,b\}} \dot{Y}_I$ and the following

$$\begin{aligned} \chi(\partial Y_{a,b}) &= \sum_{I \text{ refines } \{a,b\}, I \neq \{a,b\}} \chi(\dot{Y}_I) \\ &= \sum_{I \text{ refines } \{a,b\}, I \neq \{a,b\}} (-1)^{d(I)}\chi(Y_I) \\ &= \sum_{a<c_1<\cdots<c_s<b, s\geq 1} (-1)^{d(b)-d(a)-s-1}\chi(Y_{a,c_1})\cdots\chi(Y_{c_s,b}) \\ &= \sum_{a<c_1<\cdots<c_s<b, s\geq 1} (-1)^{d(a)+d(c_1)+\cdots+d(c_s)}\chi(a, c_1)\cdots\chi(c_s, b), \end{aligned} \tag{45}$$

where in the last step, we use the fourth and the fifth properties of $Y_{*,*}$ in Lemma 4.1.1. To further compute (45), we introduce the following functions

$$\chi_L(a,b) = \begin{cases} (-1)^{d(a)}\chi(a,b) & \text{if } a < b \\ 0 & \text{if } a = b, \end{cases} \tag{46}$$

$$\chi_R(a,b) = \begin{cases} (-1)^{d(b)}\chi(a,b) & \text{if } a < b \\ 0 & \text{if } a = b. \end{cases} \tag{47}$$

By making use of the product in the incidence algebra (see [19], for example), the right side of (45) is simply $(\chi_L^2 + \chi_L^3 + \chi_L^4 + \cdots)(a,b)$.

Proposition 4.1.3 *The condition* (42) *is equivalent to*

$$\chi_L + \chi_L^2 + \chi_L^3 + \chi_L^4 + \cdots = -\chi_R. \tag{48}$$

Proof: (48) may be rewritten as

$$1 - \chi_R = 1 + \chi_L + \chi_L^2 + \chi_L^3 + \chi_L^4 + \cdots = \frac{1}{1-\chi_L},$$

where the unit 1 of the incidence algebra of of $\tilde{P}$ is the δ-function. This is further equivalent to

$$(1-\chi_R)(1-\chi_L) = 1,$$

or

$$\chi_L + \chi_R = \chi_R\chi_L. \tag{49}$$

For $a < b$, we have

$$\chi_L(a,b) + \chi_R(a,b) = (-1)^{d(a)}\chi(a,b) + (-1)^{d(b)}\chi(a,b)$$

on the left side, and

$$\begin{aligned} \chi_R\chi_L(a,b) &= \sum_{a<c<b} \chi_R(a,c)\chi_L(c,b) \\ &= \sum_{a<c<b} (-1)^{d(c)}\chi(a,c)(-1)^{d(c)}\chi(c,b) \\ &= \sum_{a<c<b} \chi(a,c)\chi(c,b) \end{aligned}$$

on the right side. For $a = b$, both sides are zero. Therefore we see that (49) is equivalent to (42).

$\square$

Applying Proposition 4.1.3 to (45), we obtain

$$\chi(\partial Y_{a,b}) = -\chi_L(a,b) - \chi_R(a,b) = -((-1)^{d(a)} + (-1)^{d(b)})\chi(a,b).$$

Thus the required condition $\chi(Y_{a,b}) = (-1)^{d(b)-1}\chi(a,b)$ implies

$$\chi(\partial Y_{a,b}) = (1-(-1)^{d(b)-d(a)-1})\chi(Y_{a,b}).$$

The existence of the Eulerian manifold $Y_{a,b}$ with the prescribed boundary $\partial Y_{a,b}$ and the prescribed Euler characteristic $(-1)^{d(b)-1}\chi(a,b)$ then follows from the following result.

Proposition 4.1.4 *Suppose Z is an Eulerian manifold of dimension $n-1 \geq 1$ without boundary, and χ is an integer. Then there is an Eulerian manifold Y of dimension n with boundary Z and Euler characteristic χ if and only if $\chi(Z) = (1-(-1)^n)\chi$.*

Proof: If n is odd, then we may find a closed $(n-1)$-dimensional manifold M with Euler characteristic $1-\chi$. Now $Z \bigsqcup M \bigsqcup M$ has Euler characteristic $\chi(Z)+2\chi(M) = 2$. Therefore $\mathrm{cone}(Z \bigsqcup M \bigsqcup M)$ is an Eulerian manifold with boundary $Z \bigsqcup M \bigsqcup M$, and $Y = \mathrm{cone}(Z \bigsqcup M \bigsqcup M) \cup_{M\times\{0,1\}} M \times [0,1]$ is an n-dimensional Euler manifold with boundary Z. Because Y is odd dimensional, we have $2\chi(Y) = \chi(\partial Y) = \chi(Z) = 2\chi$, so that $\chi(Y) = \chi$.

If n is even, then $\chi(Z)$ always vanishes, and $\mathrm{cone}(Z)$ is an n-dimensional Eulerian manifold with boundary Z. Since n is even, we are able to find an n-dimensional closed manifold M with Euler characteristic $\chi - 1$. Then $Y = \mathrm{cone}(Z) \bigsqcup M$ is an Eulerian manifold with boundary Z and Euler characteristic χ.

□

Thus we have completed the induction on the construction of $Y_{a,b}$. It is easy to see that the five conditions on Y_I are still satisfied with the additional piece. This completes the proof of Lemma 4.1.1.

□

4.2 Construction of Eulerian Stratified Polyhedron

In this section we use Lemma 4.1.1 to prove the sufficiency part of Theorem 1.3.7, which we state again as the following lemma. We will continue using the notations introduced at the beginning of the last section.

Lemma 4.2.1 *Suppose $d(b) - 2 \geq d(a) \geq 1$ for $a < b$ in P, and (9), (14), (15) are satisfied. Then there is an Eulerian stratified polyhedron X with prescribed index set P, dimension function d, and Eulerian characteristics $\chi(a)$ and $\chi(a,b)$.*

Proof: X is constructed as the geometrical realization of a simplicial space modeled on the order complex of the partially ordered set P.

Let $I \subset J \subset P$. Then we have natural inclusion

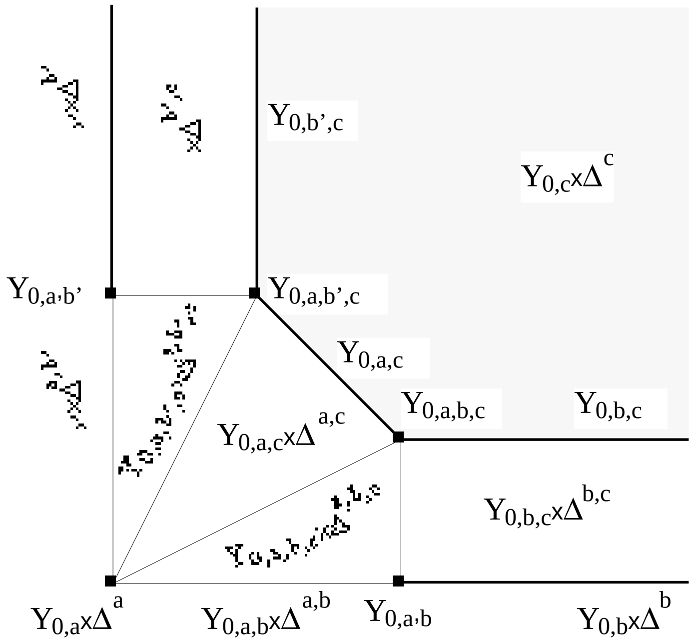
$$i_{I,J} : \Delta^I \xrightarrow{\text{incl}} \Delta^J.$$

Let $a = t(I)$ be the terminal index of I. Then $a \in J$ and $Y_{0J} = Y_{0J_{\le a}} \times Y_{J_{\ge a}}$, where

$$J_{\le a} = \{b \in J : b \le a\}, \quad J_{\ge a} = \{b \in J : b \ge a\}.$$

Since $0J_{\le a}$ refines $0I$, we have $Y_{0J_{\le a}} \subset Y_{0I}$. Denote

$$p_{I,J} : Y_{0J} \xrightarrow{\text{proj}} Y_{0J_{\le a}} \xrightarrow{\text{incl}} Y_{0I}.$$



Constructing X for P={a,b,b',c} with a<b<c, a<b'<c

It is easy to see that if $I \subset J \subset K$, then $i_{J,K} i_{I,J} = i_{I,K}$ and $p_{I,J} p_{J,K} = p_{I,K}$ (i.e., the simplices Δ^I and inclusions form a category, and $(Y_{0I}, p_{J,I})$ is a contravariant functor from this category to the category of polyhedra). Then we may form the *geometric realization* of the system (called simplicial space):

$$X = \bigcup_{I \subset P} Y_{0I} \times \Delta^I / \sim,$$

where the relation $\sim$ is given by:

$$(p_{I,J}(y), s) \sim (y, i_{I,J}(s)), \quad y \in Y_{0J}, s \in \Delta^I.$$

The image of $Y_{0I} \times \Delta^I$ in X is a closed subpolyhedron of X and will be denoted $im(Y_{0I} \times \Delta^I)$.

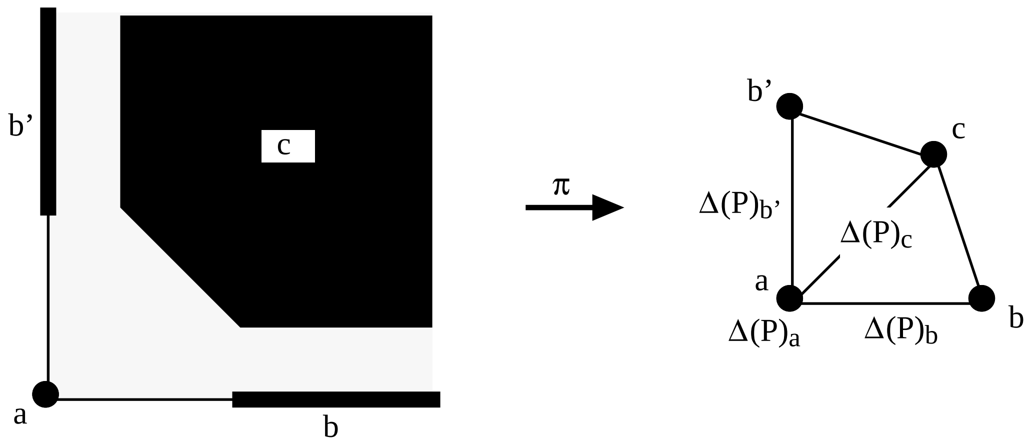
We note that if we collapse the Y_{0I} factor to a point in the construction of X, then we obtain geometric realization

$$\Delta(P) = \bigcup_{I \subset P} \Delta^I / \sim \tag{50}$$

of the order complex of P (the r-simplices of the complex are the ordered sequences of $r+1$ indices). Moreover, there is a natural projection

$$\pi : X \to \Delta(P)$$

obtained by simply forgetting the first factor (the map $Y_{0I} \to \mathit{point}$ is a natural transformation from the functor $(Y_{0I}, p_{J,I})$ to the trivial functor $(\mathit{point}, \mathit{id})$).



The projection π for P={a,b,b',c} with a<b<c, a<b'<c'

$\Delta(P)$ is naturally stratified with $\overline{\Delta(P)}_a = \Delta(P_{\leq a})$, where $P_{\leq a}$ is the partially ordered subset of P consisting of indices $\leq a$. Equivalently, this means

$$\Delta(P)_a = \bigsqcup_{I \subset P, t(I)=a} \dot{\Delta}^I. \tag{51}$$

The stratification induces a stratification on X:

$$X_a = \bigsqcup_{I \subset P, t(I)=a} Y_{0I} \times \dot{\Delta}^I = \bigsqcup_{J \supset I, t(I)=t(J)=a} \dot{Y}_{0J} \times \dot{\Delta}^I \tag{52}$$

where the second equality follows from (39). This further gives rise to the following decompositions of X

$$X = \bigsqcup_{I \subset P} Y_{0I} \times \dot{\Delta}^I = \bigsqcup_{J \supset I, t(I)=t(J)} \dot{Y}_{0J} \times \dot{\Delta}^I. \tag{53}$$

The decomposition also implies

$$\begin{aligned} im(Y_{0K} \times \Delta^K) &= \bigsqcup_{a \in K} \bigsqcup_{\substack{I \subset K_{\leq a} \\ t(I)=a}} Y_{0K_{\leq a}} \times \dot{\Delta}^I \\ &= \bigsqcup_{a \in K} \bigsqcup_{\substack{I \subset K_{\leq a} \subset J \\ t(I)=t(J)=a}} \dot{Y}_{0J} \times \dot{\Delta}^I. \end{aligned} \tag{54}$$

Now we fix $x \in X$ and compute the Euler characteristic of various links at x. By the second decomposition in (53), we assume $x \in \dot{Y}_{0J} \times \dot{\Delta}^I$ for $I \subset J$ and $t(I) = t(J) = a$. Then by the first decomposition in (52) we have for $b \geq a$ that

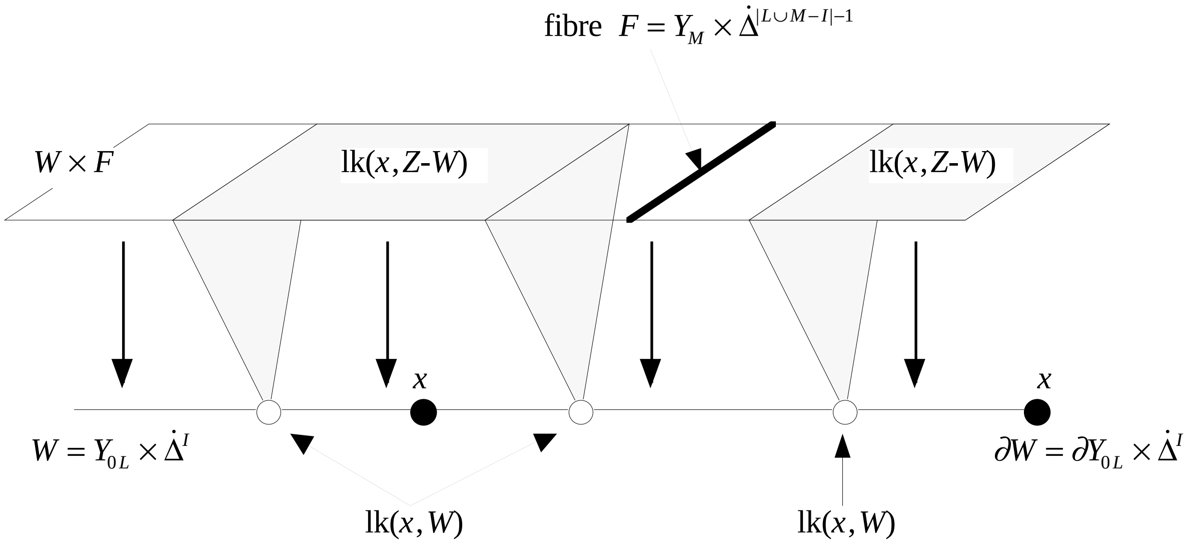
$$\mathrm{lk}(x, X_b) = \bigsqcup_{K \subset P, t(K)=b} \mathrm{lk}(x, Y_{0K} \times \dot{\Delta}^K)$$

and the Euler characteristic is simply computed by adding the Euler characteristic of the links together.

If $\mathrm{lk}(x, Y_{0K} \times \dot{\Delta}^K) \neq \emptyset$, then $x \in im(Y_{0K} \times \Delta^K)$, the closure of $Y_{0K} \times \dot{\Delta}^K$. By $x \in \dot{Y}_{0J} \times \dot{\Delta}^I$ and (54), this means $I \subset K_{\leq a} \subset J$. Denote $L = K_{\leq a}$ and $M = K_{\geq a}$. Then we have $Y_{0K} = Y_{0L} \times Y_M$ and obtain

$$\begin{aligned} \chi(\mathrm{lk}(x, X_b)) &= \sum_{K \subset P, t(K)=b, I \subset K_{\leq a} \subset J} \chi(\mathrm{lk}(x, Y_{0K} \times \dot{\Delta}^K)) \\ &= \sum_{L, M \subset P, i(M)=a, t(M)=b, I \subset L \subset J} \chi(\mathrm{lk}(x, Y_{0L} \times Y_M \times \dot{\Delta}^{L \cup M})). \end{aligned} \tag{55}$$

Abstractly, we have an Eulerian manifold $W = Y_{0L} \times \dot{\Delta}^I$ with boundary ∂W. $x \in \dot{Y}_{0J} \times \dot{\Delta}^I \subset \partial W$ when $J \neq L$ and $x \in \dot{W} = W - \partial W$ when $J = L$. We also have an Eulerian manifold $F = Y_M \times \dot{D}^{|L \cup M - I|-1}$, and the mapping cylinder $Z = W \times F \times [0,1] \cup_{\mathrm{proj}} W$ is a neighborhood of $W = Y_{0L} \times \dot{\Delta}^I$ in $(Y_{0L} \times Y_M \times \dot{\Delta}^{L \cup M}) \bigsqcup (Y_{0L} \times \dot{\Delta}^I)$. The link appearing in (55) is then $\mathrm{lk}(x, W)$ when $F = \emptyset$ (equivalent to $L = I$ and $M = \{a\}, a = b$) and $\mathrm{lk}(x, Z - W)$ in case $F \neq \emptyset$ (otherwise).



If $F \neq \emptyset$, then

$$\mathrm{lk}(x, Z - W) \cong \dot{x}\mathrm{lk}(x, W) \times F,$$

where $\dot{x}$ denotes the open cone with cone point x. Since W is an Eulerian manifold with boundary ∂W,

$$\chi(\mathrm{lk}(x, Z - W)) = \chi(\dot{x}\mathrm{lk}(x, W))\chi(F)$$

$$= \begin{cases} 0 & \text{when } x \in \partial W \\ (-1)^{\dim W}\chi(F) & \text{when } x \in \dot{W}. \end{cases}$$

If $F = \emptyset$, then by the definition of Eulerian manifold,

$$\chi(\mathrm{lk}(x, W)) = \begin{cases} 1 & \text{when } x \in \partial W \\ 1 - (-1)^{\dim W} & \text{when } x \in \dot{W}. \end{cases}$$

Translated into original notation, with $\dim W = d(0L) + |I| - 1 = d(a) - |L| + |I|$ and $\chi(F) = \chi(Y_M \times \dot{D}^{|L \cup M - I| - 1}) = (-1)^{|L\cup M - I|-1}\chi(Y_M)$, we obtain the following formula:

1. if $a = b$, then $M = \{a\}$, and

$$\chi(\mathrm{lk}(x, Y_{0L} \times \dot{\Delta}^L)) = \begin{cases} 0 & \text{when } J \neq L \neq I \\ (-1)^{d(a)-1} & \text{when } J = L \neq I \\ 1 & \text{when } J \neq L = I \\ 1 - (-1)^{d(a)} & \text{when } J = L = I. \end{cases} \tag{56}$$

2. if $a < b$, then

$$\chi(\mathrm{lk}(x, Y_{0L} \times Y_M \times \dot{\Delta}^{L \cup M})) = \begin{cases} 0 & \text{when } J \neq L \\ (-1)^{d(a)+|M|}\chi(Y_M) & \text{when } J = L. \end{cases} \tag{57}$$

Now we are ready to verify $\chi(\mathrm{lk}(x, X_b)) = \chi(a, b)$. We consider three cases.

First, if $a = b$ and $I = J$, then from (55) and (56) we have

$$\chi(\mathrm{lk}(x, X_a)) = \sum_{I \subset L \subset J} \chi(\mathrm{lk}(x, Y_{0L} \times \dot{\Delta}^L)) = 1 - (-1)^{d(a)} = \chi(a, a),$$

because we must have $J = L = I$.

Second, if $a = b$ and $I \neq J$, then again from (55) and (56)

$$\chi(\mathrm{lk}(x, X_a)) = \sum_{I \subset L \subset J} \chi(\mathrm{lk}(x, Y_{0L} \times \dot{\Delta}^L)) = 1 + (-1)^{d(a)-1} = \chi(a, a),$$

where 1 comes from choosing $L = I$, and $(-1)^{d(a)-1}$ comes from choosing $L = J$, and all other choices yield 0.

Finally, we assume $a < b$. Then to get nonzero terms in the summation we must have $L = J$. Therefore if we let $M = \{a < c_1 < \cdots < c_s < b\}$, then from (55) and (57) we have

$$\begin{aligned} \chi(\mathrm{lk}(x, X_b)) &= \sum_{a < c_1 < \cdots < c_s < b, s \geq 0} (-1)^{d(a)+s+2}\chi(Y_{a,c_1}) \cdots \chi(Y_{c_s,b}) \\ &= \sum_{a < c_1 < \cdots < c_s < b, s \geq 0} (-1)^{d(a)+d(c_1)+\cdots+d(c_s)+d(b)+1}\chi(a, c_1) \cdots \chi(c_s, b) \\ &= (-1)^{d(b)+1}(\chi_L + \chi_L^2 + \chi_L^3 + \chi_L^4 + \cdots)(a, b) \\ &= (-1)^{d(b)+1}(-\chi_R(a, b)) \\ &= \chi(a, b). \end{aligned}$$

In the fourth step we used Proposition 4.1.3.

It remains to check that $\chi(X_a)=\chi(a)$. The decomposition (52) implies

$$\begin{aligned}
\chi(X_a) &= \sum_{I,t(I)=a} \chi(Y_{0I})\chi(\dot{\Delta}^I) \\
&= \sum_{0<c_1<\cdots<c_s<a, s\geq 0} (-1)^{s+1}\chi(Y_{0,c_1})\cdots\chi(Y_{c_s,a}) \\
&= \sum_{0<c_1<\cdots<c_s<a, s\geq 0} (-1)^{d(c_1)+\cdots+d(c_s)+d(a)}\chi(c_1)\cdots\chi(c_s,a) \\
&= \sum_{0<c_1<\cdots<c_s<a, s\geq 0} (-1)^{d(0)+d(c_1)+\cdots+d(c_s)+d(a)}\chi(0,c_1)\cdots\chi(c_s,a) \\
&= (-1)^{d(a)}(\chi_L+\chi_L^2+\chi_L^3+\chi_L^4+\cdots)(0,a) \\
&= (-1)^{d(a)}(-\chi_R(0,a)) \\
&= \chi(a).
\end{aligned}$$

In the computation we used the extended notation $d(0)=-1$ and (41). In the sixth step, we used Proposition 4.1.3.

□

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