# Linear Conditions on the Number of Faces of Manifolds with Boundary

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**Abstract**

The Euler equation and the Dehn-Sommerville equations are known to be the only (rational) linear conditions for $f$-vectors (number of simplices at various dimensions) of triangulations of spheres. We generalize this fact to arbitrary triangulations, linear triangulations of manifolds, and polytopal triangulations of Euclidean balls. We prove that for closed manifolds, the Euler equation and the Dehn-Sommerville equations remain to be the only linear conditions. We also prove that for manifolds with nonempty boundary, the Euler equation is the only linear condition. These results are proved not only over $\mathbf{Q}$, but also over $\mathbf{Z}$ and $\mathbf{Z}/k\mathbf{Z}$.

# 1 Introduction

Let $X$ be an $n$-dimensional polyhedron. Associated to a triangulation $\Delta$ of $X$ is the *$f$-vector*

$$f(X;\Delta) = (f_0, f_1, \cdots, f_n),$$

where $f_i$ is the number of $i$-dimensional simplices in $\Delta$. The study of the collection of $f$-vectors of certain class of triangulations of $X$ is an important combinatorial problem. Comparatively easier problem is the determination of the affine span of these vectors, which is equivalent to finding a set of linear conditions on the $f$-vectors of $X$ so that any other such linear condition is a linear combination of these linear conditions.

Combinatorists were mostly interested in the $f$-vectors of boundaries of simplicial convex polytopes [BL, B, G1, G2, M, MS, S2]. A *convex polytope* is the convex hull of finitely many points in a Euclidean space. The boundary of a convex polytope has a natural cell complex structure, of which each cell is a convex polytope in certain supporting affine subspace. A *simplicial polytope* is a convex polytope of which the natural boundary cells are simplices. It is known for a long time that the only linear condition satisfied by $f$-vectors of boundaries of all convex polytopes is the Euler equation, and the only conditions for $f$-vectors of boundaries of all simplicial polytopes are the Euler equation and the Dehn-Sommerville equations. There are also nonlinear (including inequality) conditions. The problem of necessary and sufficient conditions for a given vector to be the $f$-vector of a simplicial polytope is completely settled in [BL2, S1].

The boundaries of simplicial polytopes are in particular triangulations of spheres. However, there are triangulations of spheres that are not boundaries of any simplicial polytopes [BW, GS]. Nevertheless, the linear conditions on all triangulations of spheres are still the Euler characteristic equation and the Dehn-Sommerville equations.

In this paper we attempt to generalize these classical results on spheres in the following directions:

1. We consider triangulations of general PL-manifolds, with or without boundary;

2. In addition to all triangulations, we also consider linear triangulations (sometimes called Euclidean simplices) with respect to specific embeddings of PL-manifolds, and polytopal triangulations of Euclidean balls;

3. In addition to rational linear conditions (which are essentially what the classical theory deals with), we consider integral and torsion linear conditions.

A triangulation $\Delta$ of a Euclidean ball $D^n$ is *polytopal* if it triangulates a polytope $P$, such that the restriction $\partial\Delta = \Delta|_{\partial P}$ is the natural cell structure of $\partial P$, and the embedding of each simplex of $\Delta$ into $P$ is linear. The fine distinction does exist here because there are triangulations of spheres that are not boundaries of any simplicial polytopes [BW, GS]. Similarly, for a PL-embedding $M \subset \mathbf{R}^N$, a triangulation $\Delta$ is called *linear* if the embedding is linear on each simplex of $\Delta$. Note that the definition depends on the choice of embedding. The fine distinction from general triangulations also exists here because there are always many nonlinear triangulations for any given

PL-embedding.

It is known that the Euler characteristic equation applies to all polyhedra, and the Dehn-Sommerville equations apply to closed manifolds. Our result in this case is no surprise: The Euler and Dehn-Sommerville equations are the only linear conditions on the $f$-vectors of triangulations of closed manifolds, regardless whether all triangulations or only linear triangulations are considered, and regardless whether the linear conditions are rational, integral, or torsion.

As for the manifolds with boundary, it is not hard to extend the Dehn-Sommerville equations. However, the meaning of these equations is quite different from the old case. It expresses the $f$-vector of the boundary in terms of the $f$-vector of the whole manifold. In fact, we may further conclude from the Dehn-Sommerville equations that the $f$-vector of the whole manifold and the $f$-vector of the interior of the manifold determine each other. As a result, we conclude the following: The Euler equation is the only linear condition on the $f$-vectors of triangulations of manifolds with nonempty boundary, regardless whether all triangulations, only linear triangulations, or only polytopal triangulations (in case the manifold is a Euclidean ball) are considered, and regardless whether the linear conditions are rational, integral, or torsion.

Another consequence of the Dehn-Sommerville equations for manifolds with boundary is that the results for closed manifolds may be extended to relative results about linear conditions on $f$-vectors of triangulations of a manifold of which the restriction on the boundary is fixed.

To state our results, we fix some notations. Given a manifold $M^n$, possibly with boundary $\partial M$, and a triangulation $\Delta$ of $M$, we may consider three $f$-

vectors:

$$\begin{array}{ll} f(M;\Delta): & \text{for whole } M \\ f(\partial M;\partial\Delta): & \text{for boundary } \partial M \\ f(M,\partial M;\Delta) = f(M;\Delta) - f(\partial M;\partial\Delta): & \text{for interior } M-\partial M \end{array}$$

where we should have added a zero to $f(\partial M;\partial\Delta)$ so that its dimension matches that of $f(M;\Delta)$. Our purpose is then to study linear conditions on

$$f(M;\text{condition}) = \left\{ f(M;\Delta): \begin{array}{c} \Delta \text{ is a triangulation of } M \\ \text{satisfying certain condition} \end{array} \right\}$$

$$f(M,\partial M;\text{condition}) = \left\{ f(M,\partial M;\Delta): \begin{array}{c} \Delta \text{ is a triangulation of } M \\ \text{satisfying certain condition} \end{array} \right\}$$

The condition may be "all", "linear", or "polytopal" in case $M$ is the Euclidean ball $D^n$. Moreover, if a triangulation $T$ of $\partial M$ extendable to the whole $M$ is fixed, we may add "rel $T$" to the condition, meaning that $\Delta|_{\partial M} = T$. Thus $f(D^n; \text{rel } \partial\Delta^n, \text{polytopal})$ means the collection of $f$-vectors of polytopal triangulations that restrict to the boundary of the standard simplex $\Delta^n$.

When considering all manifolds, we have the collection $\mathcal{M}^n$ of all $n$-dimensional PL-manifolds with boundary, and the collection $\mathcal{M}_c^n$ of all $n$-dimensional PL-manifolds without boundary. Correspondingly, we have the collections of $f$-vectors

$$\begin{aligned} f(\mathcal{M}^n;\text{condition}) = &\ \textstyle\bigcup_{M\in\mathcal{M}^n} f(M;\text{condition}) \\ f(\mathcal{M}^n,\partial\mathcal{M};\text{condition}) = &\ \textstyle\bigcup_{M\in\mathcal{M}^n} f(M,\partial M;\text{condition}) \\ f(\mathcal{M}_c^n;\text{condition}) = &\ \textstyle\bigcup_{M\in\mathcal{M}_c^n} f(M;\text{condition}) \end{aligned}$$

where "condition" may mean "all" or "linear".

The linear conditions on various collections of $f$-vectors may be expressed in terms of affine spans and linear spans. Thus we will consider the integral affine span $\mathbf{Z}$-aspan and integral linear span $\mathbf{Z}$-span. We will also consider the rational spans $\mathbf{Q}$-aspan and $\mathbf{Q}$-span.

We begin with the algebraic results on the Dehn-Sommerville equations.

**Theorem 1** *There is a universal integral $n \times (n+1)$ matrix $D(n)$, such that for any triangulation $\Delta$ of a manifold $(M^n, \partial M)$,*

$$D(n)f(M;\Delta) = -D(n)f(M, \partial M; \Delta) = f(\partial M; \partial\Delta). \tag{1}$$

*Moreover, if $D'(n)$ is the $(n+1) \times (n+1)$ matrix obtained by adding zeros as the $(n+1)$ row to $D(n)$, then*

$$\begin{cases} f(M, \partial M; \Delta) & = (I - D'(n))f(M;\Delta) \\ f(M;\Delta) & = (I - D'(n))f(M, \partial M; \Delta) \end{cases} \tag{2}$$

In case $\partial M = \emptyset$, $f(\partial M; \partial\Delta) = 0$, and (1) becomes the classical Dehn-Sommerville equation. The equation (2) explicitly expresses the $f$-vectors of the whole manifold $M$ and the interior of $M$ in terms of each other. Therefore in discussing $f$-vectors of manifolds with boundary, it suffices to only consider $f(M)$.

Now we have the two linear equations for triangulations of $(M, \partial M)$:

$$\begin{cases} D(n)f(M;\Delta) = f(\partial M; \partial\Delta) \\ \chi(f(M;\Delta)) = \chi(M) \end{cases} \tag{3}$$

where $\chi(f) = \sum(-1)^i f_i$ is the Euler characteristic. In case $\partial M = \emptyset$, or more generally, $\Delta|_{\partial M}$ is assumed to be a fixed one, the Dehn-Sommerville equations become constraints on $f(M;\Delta)$. The following is the integral version of this claim.

**Theorem 2** *Let $(M^n, \partial M)$ be a PL-manifold and let $M \subset \mathbf{R}^N$ be a PL-embedding. Given a triangulation $T$ on $\partial M$ that is extendable to a triangulation of the whole $M$ satisfying the corresponding condition, then*

$$\begin{aligned}&\mathbf{Z}\text{-aspan} f(M; \text{rel } T, \text{ all}) = \mathbf{Z}\text{-aspan} f(M; \text{rel } T, \text{ linear})\\ &\quad = \{v \in \mathbf{Z}^{n+1} : D(n)v = f(\partial M; T), \chi(v) = \chi(M)\},\end{aligned}$$

$$\begin{aligned}&\mathbf{Z}\text{-aspan} f(M, \partial M; \text{rel } T, \text{ all}) = \mathbf{Z}\text{-aspan} f(M, \partial M; \text{rel } T, \text{ linear})\\ &\quad = \{v \in \mathbf{Z}^{n+1} : D(n)v = -f(\partial M; T), \chi(v) = \chi(M)\}.\end{aligned}$$

*The characterization also applies to* $\mathbf{Z}\text{-aspan} f(D^n; \text{rel } T, \text{ polytopal})$ *and* $\mathbf{Z}\text{-aspan} f(D^n, S^{n-1}; \text{rel } T, \text{ polytopal})$.

If $\partial M \neq \emptyset$ and the triangulation on the boundary is allowed to change, then the Dehn-Sommerville equations are no longer constraints, so that the Euler equation becomes the only linear condition. The following is the integral version of this claim.

**Theorem 3** *Let $(M^n, \partial M)$ be a PL-manifold with $\partial M \neq \emptyset$, and let $M \subset \mathbf{R}^N$ be a PL-embedding. Then*

$$\begin{aligned}&\mathbf{Z}\text{-aspan} f(M; \text{all}) = \mathbf{Z}\text{-aspan} f(M; \text{linear})\\ &\quad = \{v \in \mathbf{Z}^{n+1} : \chi(v) = \chi(M)\},\end{aligned}$$

$$\mathbf{Z}\text{-aspan} f(M, \partial M; \text{all}) = \mathbf{Z}\text{-aspan} f(M, \partial M; \text{linear})$$
$$= \{v \in \mathbf{Z}^{n+1} : \chi(v) = (-1)^n \chi(M)\}.$$

*The characterization also applies to* $\mathbf{Z}\text{-aspan} f(D^n; \text{polytopal})$ *and* $\mathbf{Z}\text{-aspan} f(D^n, S^{n-1}; \text{polytopal})$*.*

The Euler equation and the Dehn-Sommerville equations are primarily linear conditions over $\mathbf{Z}$. They can be reduced to linear conditions over $\mathbf{Z}/k\mathbf{Z}$. Our claim on the torsion linear conditions means the following.

**Theorem 4** *For a manifold $M$ without boundary, or with boundary and a prescribed triangulation $T$ on $\partial M$ extendable to a triangulation of $M$ satisfying the corresponding condition, any $\mathbf{Z}/k\mathbf{Z}$-linear condition satisfied by* $f(M; \text{rel } T, \text{all})$ *or* $f(M; \text{rel } T, \text{linear})$ *is a $\mathbf{Z}/k\mathbf{Z}$-linear combination of the Euler equation and the Dehn-Sommerville equations. For a manifold $M$ with nonempty boundary, any $\mathbf{Z}/k\mathbf{Z}$-linear condition satisfied by* $f(M; \text{all})$*,* $f(M; \text{linear})$*, or* $f(D^n; \text{polytopal})$ *is a $\mathbf{Z}/k\mathbf{Z}$ multiple of the Euler equation.*

As an obvious consequence of Theorems 2 and 3, we also obtain the following characterization of $f$-vectors of all manifolds.

**Corollary 5**

$$\mathbf{Z}\text{-aspan} f(\mathcal{M}^n) = \mathbf{Z}\text{-aspan} f(\mathcal{M}^n, \partial\mathcal{M}) = \mathbf{Z}^{n+1},$$

$$\mathbf{Z}\text{-aspan} f(\mathcal{M}^n_c) = \ker D(n).$$

Moreover, similar statements may be made on torsion linear conditions on $f$-vectors over all manifolds.

The corollary in particular means that there is no nontrivial linear condition satisfied by the $f$-vectors of all manifolds. However, we always have

$$D(n-1)D(n)f(M;\Delta) = D(n-1)f(\partial M;\partial\Delta) = 0$$

because $\partial\partial M = \emptyset$. From algebraic topology, we also have

$$\chi(D(n)f(M;\Delta)) = \chi(f(\partial M;\partial\Delta)) = \begin{cases} 0 & \textit{for even } n \\ 2\chi(f(M;\Delta)) & \textit{for odd } n \end{cases}$$

By plugging the first equation of (2) into the second, we again obtain a linear equation

$$(I - D'(n))^2 f(M;\Delta) = f(M;\Delta).$$

These are all universal linear equations, regardless of the choice of $M$. Therefore we obtain the following algebraic formulas.

**Corollary 6**

$$D(n-1)D(n) = 0, \tag{4}$$

$$\chi(D(n)v) = \begin{cases} 0 & \text{for even } n \\ 2\chi(v) & \text{for odd } n \end{cases}, \tag{5}$$

$$(I - D'(n))^2 = I. \tag{6}$$

The identity (4) gives rise to the following *Dehn-Sommerville chain complex*:

$$DS_* : \cdots \longrightarrow \mathbf{Z}^{n+1} \xrightarrow{D(n)} \mathbf{Z}^n \longrightarrow \cdots \longrightarrow \mathbf{Z}^2 \xrightarrow{D(1)} \mathbf{Z},$$

where $DS_n = \mathbf{Z}^{n+1}$. We may compare it with the geometrical "chain complex"

$$\mathcal{M}^* : \cdots \longrightarrow \mathcal{M}^{n+1} \xrightarrow{\partial} \mathcal{M}^n \longrightarrow \cdots \longrightarrow \mathcal{M}^1 \xrightarrow{\partial} \mathcal{M}^0.$$

The Dehn-Sommerville equation (3) means that the map

$$f\text{-vector} : \mathcal{M}^* \to DS_*$$

is a "chain homomorphism". In particular, any closed triangulated $n$-dimensional manifold produces a homology class in $H_n DS$.

**Theorem 7**

$$H_n DS \cong \begin{cases} \mathbf{Z}_2 & \textit{for even } n \\ 0 & \textit{for odd } n \end{cases}$$

*The isomorphism is given by the Euler characteristic in the following sense:*

1. *For* $[f] \in H_{odd} DS$, $\chi(f) = 0$;
2. *For* $[f] \in H_{even} DS$, $\chi(f) = 0$ *modulo* 2 *if and only if* $[f] = 0$.

The paper is organized as follows. We first derive the Dehn-Sommerville equations for the manifolds with boundary and prove Theorem 1. Then we establish the existence of some special polytopal triangulations which are needed to prove our theorems on linear conditions. After that we prove the rational version (i.e., the classical type results) of Theorems 2 and 3. Logically this is sufficient for us to conclude the rational version of Corollary 5, which in turn implies Corollary 6. With the aid of Corollary 6, we discuss the Dehn-Sommerville homology. The result is then used to prove Theorems 2 and 3 about integral linear conditions on $f$-vectors. Finally the Dehn-Sommerville homology is used again to prove Theorem 4 about torsion linear conditions.

## 2 Dehn-Sommerville Equations

Let $M^n$ be a manifold with boundary $\partial M$ and let $\Delta$ be a (combinatorial) triangulation. For $0 \le i \le n-1$, the link $\mathrm{lk}(\sigma, \Delta)$ of a simplex $\sigma^i \in \Delta$ is $S^{n-i-1}$ when $\sigma$ is in the interior of $M$, and is $D^{n-i-1}$ when $\sigma$ is in the boundary of $M$. Since $f_i(M, \partial M; \Delta)$ $(= f_i(M; \Delta) - f_i(\partial M; \partial\Delta))$ and $f_i(\partial M; \partial\Delta)$ are respectively the numbers of $i$-simplices in the interior and in the boundary of $M$, we have the equality:

$$\begin{aligned}\sum_{\dim\sigma=i} \chi(\mathrm{lk}(\sigma,\Delta)) &= \chi(S^{n-i-1})(f_i(M;\Delta) - f_i(\partial M;\partial\Delta)) \\ &\quad + \chi(D^{n-i-1}) f_i(\partial M; \partial\Delta). \end{aligned} \tag{7}$$

As in the case of closed manifolds, the left side may be computed in terms of $f_j(M;\Delta)$, $j > i$. The following detail essentially follows from [K]:

$$\begin{aligned}\sum_{\dim\sigma=i} \chi(\mathrm{lk}(\sigma,\Delta)) &= \sum_{\dim\sigma=i} \sum_{\substack{\tau\cap\sigma=\emptyset \\ \tau \text{ and } \sigma \text{ form a simplex}}} (-1)^{\dim\tau} \\ &= \sum_{\dim\sigma=i} \sum_{\substack{\rho\supset\sigma \\ \rho\neq\sigma}} (-1)^{\dim\rho-i-1} \\ &= \sum_{\dim\rho>i} \sum_{\substack{\sigma\subset\rho \\ \dim\sigma=i}} (-1)^{\dim\rho-i-1} \\ &= \sum_{\dim\rho>i} (-1)^{\dim\rho-i-1} \binom{\dim\rho+1}{i+1} \\ &= \sum_{j>i} (-1)^{j-i-1} \binom{j+1}{i+1} f_j(M;\Delta). \end{aligned}$$

Then we obtain the following Dehn-Sommerville equations for manifolds with boundary.

**Lemma 8** *The $f$-vector of the boundary of a manifold is determined by the $f$-vector of the whole manifold as the following*

$$f_i(\partial M;\partial\Delta) = (1-(-1)^{n-i})f_i(M;\Delta) + \sum_{j=i+1}^{n} (-1)^{n-j-1}\binom{j+1}{i+1} f_j(M;\Delta). \tag{8}$$

Denote by $D(n)$ the $n\times(n+1)$ matrix of the coefficients of the right side of (8). The rank of $D(n)$ is known to be $[\frac{n+1}{2}]$ (see [K], for example). It is also known that for even $n$, the Euler characteristic is independent of the rows of $D(n)$, and for odd $n$, the Euler characteristic is a linear combination of the rows of $D(n)$. In fact, for odd $n$, the identity (5) of Corollary 6 explicitly expresses the Euler characteristic as one-half of the alternating sum of the rows of $D(n)$.

$D(n)$ is explicitly given as follows:

*for even $n$:*

$$D(n) = \begin{pmatrix} 0 & \binom{2}{1} & -\binom{3}{1} & \binom{4}{1} & -\binom{5}{1} & \cdots & \binom{n}{1} & -\binom{n+1}{1} \\ 0 & 2 & -\binom{3}{2} & \binom{4}{2} & -\binom{5}{2} & \cdots & \binom{n}{2} & -\binom{n+1}{2} \\ 0 & 0 & 0 & \binom{4}{3} & -\binom{5}{3} & \cdots & \binom{n}{3} & -\binom{n+1}{3} \\ 0 & 0 & 0 & 2 & -\binom{5}{4} & \cdots & \binom{n}{4} & -\binom{n+1}{4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \binom{n}{n-1} & -\binom{n+1}{n-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -\binom{n+1}{n} \end{pmatrix}$$

*for odd $n$*:

$$D(n) = \begin{pmatrix} 2 & -\binom{2}{1} & \binom{3}{1} & -\binom{4}{1} & \binom{5}{1} & \cdots & \binom{n}{1} & -\binom{n+1}{1} \\ 0 & 0 & \binom{3}{2} & -\binom{4}{2} & \binom{5}{2} & \cdots & \binom{n}{2} & -\binom{n+1}{2} \\ 0 & 0 & 2 & -\binom{4}{3} & \binom{5}{3} & \cdots & \binom{n}{3} & -\binom{n+1}{3} \\ 0 & 0 & 0 & 0 & \binom{5}{4} & \cdots & \binom{n}{4} & -\binom{n+1}{4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \binom{n}{n-1} & -\binom{n+1}{n-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -\binom{n+1}{n} \end{pmatrix}$$

*Proof of Theorem 1*: In terms of the matrices given above, the Dehn-Sommerville equations (8) may be interpreted as

$$f(\partial M; \partial\Delta) = D(n) f(M; \Delta).$$

To prove $D(n)f(M;\Delta) = -D(n)f(M, \partial M; \Delta)$, we consider the double $M \cup_{\partial M} M$ with the triangulation $\Delta \cup_{\partial\Delta} \Delta$. Since $M \cup_{\partial M} M$ is a closed manifold, $\Delta \cup_{\partial\Delta} \Delta$ satisfies the classical Dehn-Sommerville equations:

$$D(n) f(M \cup_{\partial M} M; \Delta \cup_{\partial\Delta} \Delta) = 0. \tag{9}$$

However, we clearly have

$$\begin{aligned} f(M \cup_{\partial M} M; \Delta \cup_{\partial\Delta} \Delta) = & \;\; 2f(M;\Delta) - f(\partial M; \partial\Delta) \\ = & \;\; f(M;\Delta) + f(M, \partial M; \Delta). \end{aligned} \tag{10}$$

Putting (10) into (9), we conclude that

$$D(n)f(M;\Delta) + D(n)f(M, \partial M; \Delta) = 0.$$

This completes the proof of (1).

The equalities in (2) follow easily from (1):

$$\begin{aligned}
f(M,\partial M;\Delta) = &\ f(M;\Delta) - (f(\partial M;\partial\Delta),0) \\
= &\ f(M;\Delta) - (D(n)f(M;\Delta),0) \\
= &\ (I - D'(n))f(M;\Delta); \\
f(M;\Delta) = &\ f(M,\partial M;\Delta) + (f(\partial M;\partial\Delta),0) \\
= &\ f(M,\partial M;\Delta) - (D(n)f(M,\partial M;\Delta),0) \\
= &\ (I - D'(n))f(M,\partial M;\Delta).
\end{aligned}$$

# 3 Affine Independent Cyclic Polytopes

In this part we find triangulations $\delta_0, \delta_1, \cdots, \delta_{[\frac{n+1}{2}]}$ of $(D^n, S^{n-1})$ with the following properties:

1. $\partial\delta_i = \partial\Delta^n$ is the standard triangulation of $S^{n-1}$, the boundary of the simplex $\Delta^n$;
2. $\delta_i$ is polytopal;
3. The differences
$$f(D^n;\delta_i) - f(D^n;\delta_0), \quad 1 \le i \le [\tfrac{n+1}{2}] \tag{11}$$
are linearly independent and integrally span a direct summand of $\mathbf{Z}^{n+1}$.

In other words, we have vectors from $f(D^n; \mathrm{rel}\ \partial\Delta^n$, polytopal) that integrally span an affine space of dimension $[\frac{n+1}{2}]$ with the parallel linear lattice to be a direct summand.

The proof of the rational versions of Theorems 2 and 3 for all triangulations is based on the first property and the affine independence of $f(D^n;\delta_i)$,

$0 \le i \le [\frac{n+1}{2}]$, which is a consequence of the third property. The second property is needed for the proof on linear and polytopal triangulations. The direct summand requirement in the third property is needed to further prove our Theorems 2, 3, and 4 on integral and torsion linear conditions.

Suppose that $\delta$ is a triangulation of $S^n$. If we delete the interior of an $n$-dimensional simplex from $\delta$, then we obtain a triangulation of $D^n$ that restricts to $\partial\Delta^n$ on $S^{n-1}$. The problem is to find a good $\delta$ so that this triangulation is polytopal.

We recall the definition of cyclic polytopes from [B]. For $p > n \ge 2$, let $t_1 < t_2 < \cdots < t_p$, and

$$x(t) = (t, t^2, \cdots, t^n) \in \mathbf{R}^n$$

$$C(p,n) = \text{convex hull}\{x(t_1), x(t_2), \cdots, x(t_p)\}.$$

It is proved in [B] that $C(p,n)$ is a simplicial polytope of dimension $n$, and up to affine equivalence, is independent of the choice of the numbers $t_i$. Moreover, the $f$-vectors are of the form

$$f(C(p,n)) = \left(\binom{p}{1}, \binom{p}{2}, \cdots, \binom{p}{[\frac{n}{2}]}, f_{[\frac{n}{2}]}(C(p,n)), \cdots, 1\right), \tag{12}$$

where $f_n(C(p,n)) = 1$ denotes the fact that $C(p,n)$ has only one top dimensional cell.

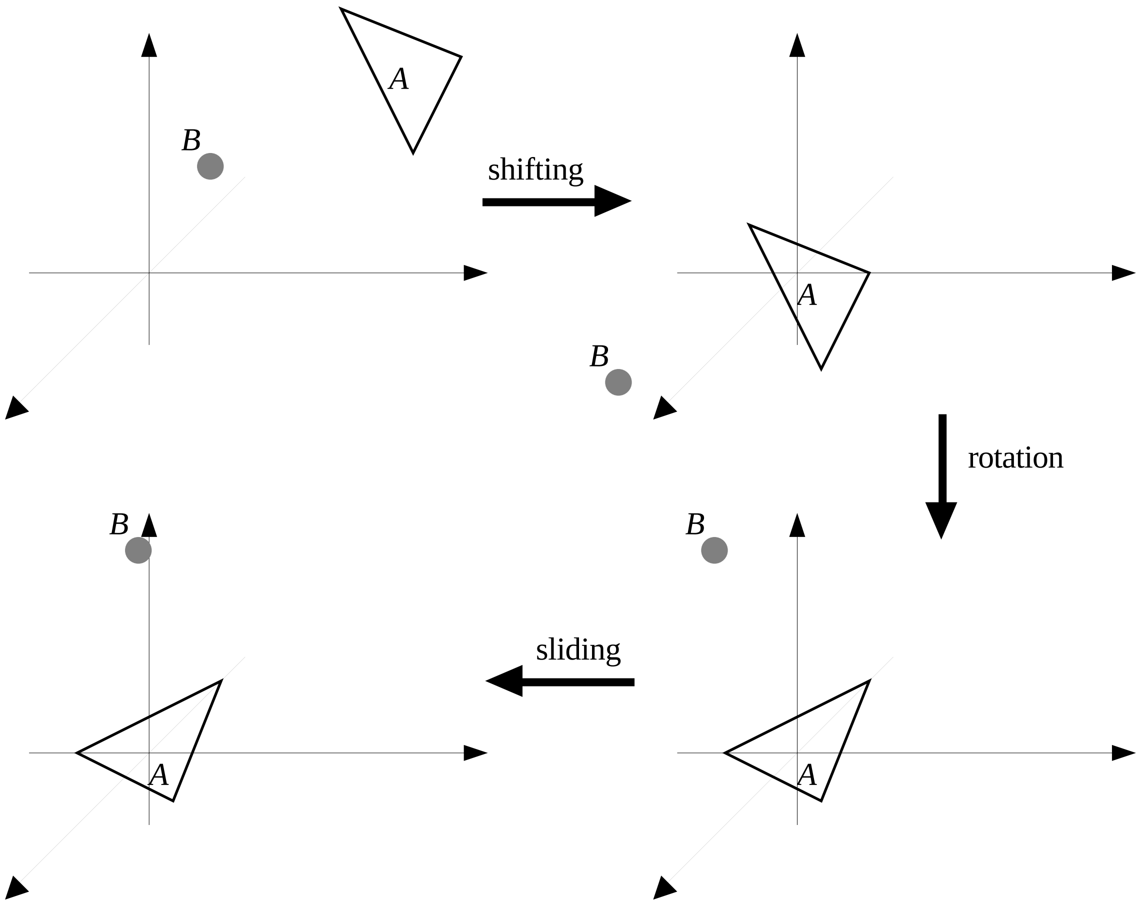
The boundary of $C(p,n+1)$ ($p \ge n+2$) is a triangulation of $S^n$. Gale's evenness condition (see page 87 of [B]) says that a collection $X \subset \{x(t_1), x(t_2), \cdots, x(t_p)\}$ of $n+1$ points is the vertex set of a simplex of the triangulation if and only if any proper maximal segment $Y = \{x(t_i), x(t_{i+1}), \cdots, x(t_j)\} \subset X$ consists of even number of points. Here the properness means that the first and the last vertices $x(t_1)$ and $x(t_p)$ are not in $Y$. The maximality means that $x(t_{i-1})$ and $x(t_{j+1})$ are not in $X$.

The collection $\{x(t_1), x(t_2), \cdots, x(t_{n+1})\}$ is not proper and therefore form an $n$-dimensional simplex $A$ of the triangulation $\partial C(p, n+1)$ of $S^n$. By deleting the interior of $A$ from $\partial C(p, n+1)$, we obtain a triangulation $D(p, n)$ of $D^n \cong S^n - \text{int}A$. The restriction of $D(p, n)$ on the boundary is the same as the boundary of $A$. Hence $D(p, n)|_{S^{n-1}} = \partial\Delta^n$.

**Lemma 9** *$D(p, n)$ is a polytopal triangulation.*

*Proof*: $A$ is the $n$-simplex spanned by the first $n+1$ vertices. Let $B$ be the collection of the last $p-n-1$ vertices. By taking $t_{n+2}, t_{n+3}, \cdots, t_p$ very close to each other, we may assume that the size of $B$ is very small compared with the size of $A$.

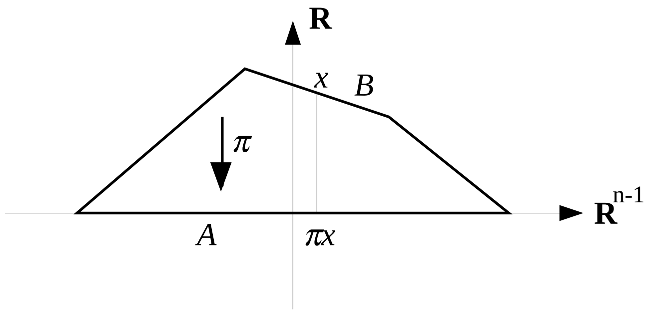
We may shift $C(p, n+1)$ so that the center of $A$ goes to the origin of $\mathbf{R}^{n+1}$. Then we may rotate $C(p, n+1)$ so that $A$ completely lies in $\mathbf{R}^n \times 0$. After that we may slide (keeping $\mathbf{R}^n \times 0$ fixed) so that the projection $\pi : \mathbf{R}^{n+1} \to \mathbf{R}^n \times 0$ carries $B$ to the interior of $A$. This is always possible for small enough $B$.




$A \quad$ = simplex with vertices $x(t_1), \cdots, x(t_{n+1})$

$B \quad$ = vertices $x(t_{n+2}), \cdots, x(t_p)$

The transformations we have done are all affine equivalences. Therefore $C(p, n+1)$ is still a simplicial polytope after the transformations.




Since $\pi$ maps $B$ to the interior of $A$, it maps the whole polytope $C(p, n+1)$ onto $A$. We claim that the restriction on $D(p, n)$ is a one-to-one correspondence. In fact, for any $x$ in the interior of $D(p, n)$, $x$ and $\pi x$ belong to different facets of $C(p, n+1)$. By the convexity of $C(p, n+1)$, $x$ and $\pi x$ are the only boundary points on the line segment $[x, \pi x] = \pi^{-1}(\pi x) \cap C(p, n+1)$. Consequently, $x$ is the only point in $D(p, n)$ that is mapped to $\pi x$ by $\pi$. This proves injectivity. The surjectivity follows from the fact that both $D(p, n)$ and $A$ are triangulations of $D^n$, and the map $\pi$ is identity on the boundary.

$\pi$ is a linear map, therefore it realizes $D(p, n)$ as a polytopal triangulation of $A$. This completes the proof of the Lemma.

The $f$-vector of $D(p, n)$ can be easily obtained from the $f$-vector of $C(p, n+1)$ (see (12)):

$$\begin{aligned} & f(D(p,n)) \\ = \ & \left(\binom{p}{1}, \binom{p}{2}, \cdots, \binom{p}{[\frac{n+1}{2}]}, f_{[\frac{n+1}{2}]}(C(p, n+1)), \cdots, f_n(C(p, n+1)) - 1\right). \end{aligned} \tag{13}$$

As pointed out before, we need the following Lemma for our result on the torsion conditions.

**Lemma 10** *For any $p \geq n+2$, the vectors*

$$f(D(p+i,n)) - f(D(p,n)), \quad 1 \leq i \leq [\tfrac{n+1}{2}] \tag{14}$$

*integrally generate a direct summand of $\mathbf{Z}^{n+1}$.*

*Proof*: The lattice of $\mathbf{Z}^{n+1}$ integrally generated by (14) is equivalent to the lattice integrally generated by

$$\begin{aligned} & f(D(p+i,n)) - f(D(p+i-1,n)) \\ = \ & \left(\binom{p+i-1}{0}, \binom{p+i-1}{1}, \cdots, \binom{p+i-1}{l-1}, \cdots, \right), \quad 1 \leq i \leq l = [\tfrac{n+1}{2}] \end{aligned} \tag{15}$$

where we make use of the equality $\binom{p+i}{j} - \binom{p+i-1}{j} = \binom{p+i-1}{j-1}$. Thus the matrix formed by (15) is of the form $(A_l^p, *)$ with

$$A_l^p = \begin{pmatrix} \binom{p}{0} & \binom{p}{1} & \binom{p}{2} & \cdots & \binom{p}{l-1} \\ \binom{p+1}{0} & \binom{p+1}{1} & \binom{p+1}{2} & \cdots & \binom{p+1}{l-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{p+l-1}{0} & \binom{p+l-1}{1} & \binom{p+l-1}{2} & \cdots & \binom{p+l-1}{l-1} \end{pmatrix}$$

By subtracting $(i-1)$ row from $i$ row, we obtain

$$\det A_l^p = \det \begin{pmatrix} \binom{p}{0} & \binom{p}{1} & \binom{p}{2} & \cdots & \binom{p}{l-1} \\ 0 & \binom{p}{0} & \binom{p}{1} & \cdots & \binom{p}{l-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \binom{p+l-2}{0} & \binom{p+l-2}{1} & \cdots & \binom{p+l-2}{l-2} \end{pmatrix} = \det A_{l-1}^p.$$

By induction we see that $\det A_l^p = \det A_1^p = 1$. Consequently, $A_l^p$ is integrally invertible.

As a result, we see that the vectors (15) and $e_j = (0, 0, \cdots, 1_{(j)}, \cdots, 0)$, $l + 1 \leq j \leq n + 1$ form an integral basis of $\mathbf{Z}^{n+1}$. This completes the proof of Lemma 10.

# 4 Rational Linear Conditions

With polytopal triangulations $\delta_i$ constructed in the last section, we are able to prove the rational version of Theorem 2. The idea is to show that various subdivisions of a top simplex of some triangulation provides enough variations on the triangulations so that the affine span of the corresponding $f$-vectors has dimension no smaller than the dimension of the affine space characterized by the Dehn-Sommerville equations and the Euler characteristic equation.

**Theorem 11** *Let $(M^n, \partial M)$ be a PL-manifold and let $M \subset \mathbf{R}^N$ be a PL-embedding. Given a triangulation $T$ on $\partial M$ that is extendable to a triangulation of the whole $M$ satisfying the corresponding condition. Then*

$$\begin{aligned}\mathbf{Q}\text{-aspan} f(M; \mathrm{rel}\ T,\ \mathrm{all}) &= \mathbf{Q}\text{-aspan} f(M; \mathrm{rel}\ T,\ \mathrm{linear}) \\ &= \{v \in \mathbf{Q}^{n+1} : D(n)v = f(\partial M; T), \chi(v) = \chi(M)\},\end{aligned}$$

$$\begin{aligned}\mathbf{Q}\text{-aspan} f(M, \partial M; \mathrm{rel}\ T,\ \mathrm{all}) &= \mathbf{Q}\text{-aspan} f(M, \partial M; \mathrm{rel}\ T,\ \mathrm{linear}) \\ &= \{v \in \mathbf{Q}^{n+1} : D(n)v = -f(\partial M; T), \chi(v) = \chi(M)\}.\end{aligned}$$

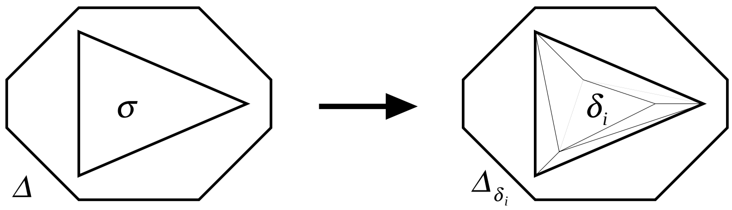
*The characterization also applies to* $\mathbf{Q}$-aspan$f(D^n; \text{rel } T, \text{ polytopal})$ *and* $\mathbf{Q}$-aspan$f(D^n, S^{n-1}; \text{rel } T, \text{ polytopal})$.

*Proof*: We fix a triangulation $\Delta$ of $M$, such that $\Delta|_{\partial M} = T$. We also fix a top simplex $\sigma^n$ of $\Delta$. In case we are considering linear or polytopal triangulations, we assume $\Delta$ is linear or polytopal.

We take the special polytopal triangulations $\delta_i$ constructed in the last section and replace $\sigma$ by $\delta_i$ to obtain triangulations

$$\Delta_{\delta_0}, \Delta_{\delta_1}, \cdots, \Delta_{\delta_{[\frac{n+1}{2}]}}. \tag{16}$$

Since $\delta_i$ are polytopal, the linearity or the polytopal property is preserved after the replacement.




The $f$-vectors are changed as follows

$$f(M; \Delta_{\delta_i}) = f(M; \Delta) - f(S^{n-1}; \partial\Delta^n) + f(D^n; \delta_i). \tag{17}$$

Since $f(D^n; \delta_i)$, $0 \leq i \leq [\frac{n+1}{2}]$, are assumed to be affine independent, the $f$-vectors $f(M; \Delta_{\delta_i})$ are also affine independent. Therefore the dimension of $\mathbf{Q}$-aspan$f(M; \text{rel } T, \text{ condition})$ is at least $[\frac{n+1}{2}]$. On the other hand, the affine subspace characterized by the Euler equation and the Dehn-Sommerville equations is known to be also $[\frac{n+1}{2}]$. Consequently the two affine spaces coincide.

Next we turn to the rational version of Theorem 3.

**Theorem 12** *Let $(M^n, \partial M)$ be a PL-manifold with $\partial M \neq \emptyset$, and let $M \subset \mathbf{R}^N$ be a PL-embedding. Then*

$$\begin{aligned} \mathbf{Q}\text{-aspan} f(M; \text{all}) &= \mathbf{Q}\text{-aspan} f(M; \text{linear}) \\ &= \{v \in \mathbf{Q}^{n+1} : \chi(v) = \chi(M)\}. \end{aligned}$$

$$\begin{aligned} \mathbf{Q}\text{-aspan} f(M, \partial M; \text{all}) &= \mathbf{Q}\text{-aspan} f(M, \partial M; \text{linear}) \\ &= \{v \in \mathbf{Q}^{n+1} : \chi(v) = (-1)^n \chi(M)\}. \end{aligned}$$

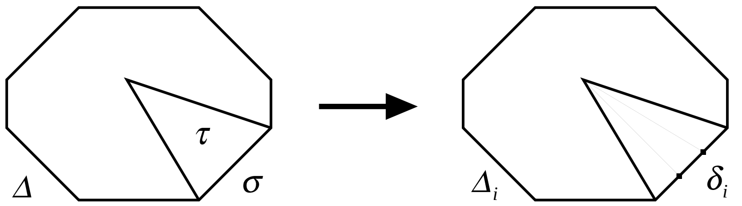
*The characterization also applies to* $\mathbf{Q}\text{-aspan} f(D^n; \text{polytopal})$ *and* $\mathbf{Q}\text{-aspan} f(D^n, S^{n-1}; \text{polytopal})$.

*Proof*: We again fix a triangulation $\Delta$ of $M$ and a top simplex $\sigma^{n-1}$ of $\partial \Delta$. In case we are considering linear triangulations, we also assume $\Delta$ is linear. As in the proof of Theorem 11, we may replace $\sigma$ by special polytopal triangulations to obtain triangulations

$$\partial \Delta_0, \partial \Delta_1, \cdots, \partial \Delta_{[\frac{n}{2}]} \tag{18}$$

with affine independent $f$-vectors.

Note that $\sigma$ is a face of a unique $n$-dimensional simplex $\tau$ of $\Delta$. Since $\partial\Delta_i$ are obtained from $\partial\Delta$ by subdividing the interior of $\sigma$, the cone construction (in the interior of $\tau$) easily extends the subdivision into a linear subdivision $\Delta_i$ of $\Delta$ with the prescribed boundary $\partial\Delta_i$. Thus we obtain triangulations $\Delta_i$, $0 \leq i \leq [\frac{n}{2}]$, of $M$ with affine independent boundary $f$-vectors.




If the linearity is required, then the construction above still produces linear triangulations. In case of polytopal triangulations of $D^n$, however, the triangultions $\Delta_i$ above are not polytopal, because subdividing the faces of polytopes produces geometrically unnatural triangulations. We may instead take $\Delta_i$ to be the cone triangulation of $C(p+i,n)$. Then the $f$-vectors of their boundaries are indeed affine independent. This is the property needed for subsequent proof.

Next we modify the interior of a top simplex of $\Delta_0$ in the same way to obtain triangulations

$$\Delta_{\delta_0}, \Delta_{\delta_1}, \cdots, \Delta_{\delta_{[\frac{n+1}{2}]}}, \tag{19}$$

with affine independent $f$-vectors. Moreover, since the modification is done in the interior of $M$, their boundaries are the same: $\partial\Delta_{\delta_j} = \partial\Delta_0$. We also note that, as in the proof of Theorem 11, the linearity or the polytopal property is preserved after the replacement.

The Theorem is proved by showing that the triangulations

$$\Delta_1, \Delta_2, \cdots, \Delta_{[\frac{n}{2}]}; \quad \Delta_{\delta_0}, \Delta_{\delta_1}, \cdots, \Delta_{\delta_{[\frac{n+1}{2}]}} \tag{20}$$

have affine independent $f$-vectors. Set

$$\sum_{i=1}^{[\frac{n}{2}]} a_i[f(M;\Delta_i) - f(M;\Delta_{\delta_0})] + \sum_{j=1}^{[\frac{n+1}{2}]} b_j[f(M;\Delta_{\delta_j}) - f(M;\Delta_{\delta_0})] = 0. \tag{21}$$

Apply the Dehn-Sommerville matrix $D(n)$ to the equation (21). By Theorem 1, we obtain

$$\sum_{i=1}^{[\frac{n}{2}]} a_i[f(\partial M;\partial\Delta_i) - f(\partial M;\partial\Delta_{\delta_0})] + \sum_{j=1}^{[\frac{n+1}{2}]} b_j[f(\partial M;\partial\Delta_{\delta_j}) - f(\partial M;\partial\Delta_{\delta_0})] = 0. \tag{22}$$

Since $\partial\Delta_{\delta_j} = \partial\Delta_0$, (22) becomes

$$\sum_{i=1}^{[\frac{n}{2}]} a_i[f(\partial M;\partial\Delta_i) - f(\partial M;\partial\Delta_{\delta_0})] = 0.$$

From our construction, the $f$-vectors of (18) are affine independent. Therefore we conclude that all $a_i = 0$, and the equation (21) becomes

$$\sum_{j=1}^{[\frac{n+1}{2}]} b_j[f(M;\Delta_{\delta_j}) - f(M;\Delta_{\delta_0})] = 0.$$

Since the $f$-vectors of (19) are also affine independent, we conclude that all $b_j = 0$.

We have found $[\frac{n}{2}] + [\frac{n+1}{2}] = n+1$ affine independent $f$-vectors of $M$. Hence the dimension of $\mathbf{Q}$-aspan$f(M;\text{condition})$ is at least $n$. On the other hand, the affine space is included in the affine hyperplane specified by the Euler equation $\chi(f) = \chi(M)$. The hyperplane has dimension $n$. Thus we conclude that the two affine spaces are the same.

# 5 Dehn-Sommerville Homology

As pointed out in the introduction, (4) of Corollary 6 gives rise to the Dehn-Sommerville chain complex. It is easy to see that $H_0DS = \mathbf{Z}_2$ and $H_1DS = 0$. The first part of Theorem 7 may be obtained by showing that the homology is 2-periodic.

Note that

$$D(n+2) = \begin{pmatrix} D(n) & * \\ 0 & E(n) \end{pmatrix},$$

where

$$E(n) = \begin{pmatrix} n+2 & -\frac{1}{2}(n+2)(n+3) \\ 2 & -(n+3) \end{pmatrix}.$$

In particular, we have the chain map:

$$\begin{array}{ccccccccccc} \Sigma^2 DS_* : & \cdots \to & \mathbf{Z}^3 & \stackrel{D(2)}{\to} & \mathbf{Z}^2 & \stackrel{D(1)}{\to} & \mathbf{Z} & \stackrel{D(0)}{\to} & 0 & \to & 0 \\ \downarrow \imath_* & & \downarrow \imath_4 & & \downarrow \imath_3 & & \downarrow \imath_2 & & \downarrow \imath_1 & & \downarrow \imath_0 \\ DS_* : & \cdots \to & \mathbf{Z}^5 & \stackrel{D(4)}{\to} & \mathbf{Z}^4 & \stackrel{D(3)}{\to} & \mathbf{Z}^3 & \stackrel{D(2)}{\to} & \mathbf{Z}^2 & \stackrel{D(1)}{\to} & \mathbf{Z} \end{array} \tag{23}$$

consisting of inclusion maps (i.e., adding two zeros at the end).

**Lemma 13** $\imath_*$ *induces equivalences*

$$H_nDS \cong H_{n+2}DS, \quad n \geq 0.$$

*Proof*: $\imath_*$ fits into a short exact sequence of chain complexes:

$$0 \to \Sigma^2 DS_* \stackrel{\imath_*}{\to} DS_* \stackrel{\pi_*}{\to} T_* \to 0,$$

where $T_*$ is the chain complex formed by the last two coordinates:

$$T_* : \cdots \longrightarrow \mathbf{Z}^2 \stackrel{E(n)}{\longrightarrow} \cdots \longrightarrow \mathbf{Z}^2 \stackrel{E(2)}{\longrightarrow} \mathbf{Z}^2 \stackrel{D(1)}{\longrightarrow} \mathbf{Z},$$

and $\pi_*$ is the projection to the last two coordinates. Therefore proving $\imath_*$ induces homological equivalence is equivalent to proving that $H_n T_* = 0$ for $n \geq 2$.

A pair of integers $(a, b) \in \ker E(n)$ means exactly $2a - (n+3)b = 0$. Showing that $(a, b) \in \mathrm{im} E(n+1)$ means exactly $a = (n+3)c - \frac{1}{2}(n+3)(n+4)d$, $b = 2c - (n+4)d$ for some integers $c, d$. In case $n = 2l - 1$ is odd, we have $a = (l+1)b$, and we may choose $c = (l+2)b$ and $d = b$. In case $n = 2l$ is even, we have $2a = (2l+3)b$, which implies that $a = (2l+3)e$ and $b = 2e$ for some integer $e$. Then we may choose $c = e$ and $d = 0$.

This proves that $\ker E(n) = \mathrm{im} E(n+1)$ and as result completes the proof of the Lemma above.

It remains to verify the relation between the Euler characteristic and the homology.

For odd $n$, an element of $H_n DS$ is represented by $f$ such that $D(n)f = 0$. Thus by (5) we have $2\chi(f) = \chi(D(n)f) = 0$. Consequently, $\chi(f) = 0$.

For even $n$, an element $[f] \in H_n DS$ is 0 if $f = D(n+1)v$ for some $v$. By (5), $\chi(f) = \chi(D(n+1)v) = 2\chi(v)$. Consequently, the $\mathbf{Z}_2$-Euler characteristic of $f$ vanishes. This shows that the $\mathbf{Z}_2$-Euler characteristic is well defined on $H_n DS$. Moreover, the nonzero element in $H_n DS$ is represented by $1_{n+1} = (1, 0, 0, \cdots, 0)$ (note that $1_{n+1} = \imath(1_{n-1}) = \imath^{\frac{n}{2}} 1_1$), whose $\mathbf{Z}_2$-Euler characteristic is nonzero. This shows that the $\mathbf{Z}_2$-Euler characteristic indeed provides the isomorphism $H_n DS \cong \mathbf{Z}_2$.

# 6 Integral Linear Conditions

In this part we prove Theorems 2 and 3 on integral linear conditions. For this purpose, we observe the following facts in the proof of Theorems 11 and 12.

If $\partial M = \emptyset$, or more generally if a triangulation $T$ of $\partial M$ has been fixed, then we obtain in the proof of Theorem 11 triangulations (16) such that $\Delta_{\delta_i}|_{\partial M} = T$ and their $f$-vectors are affine independent. As a matter of fact, it follows from (17) and Lemma 10 that we may further assume

$$f(M; \Delta_{\delta_i}) - f(M; \Delta_{\delta_0}), \quad 1 \leq i \leq [\tfrac{n+1}{2}] \tag{24}$$

integrally span a direct summand of $\mathbf{Z}^{n+1}$.

If $\partial M \neq \emptyset$ and the triangulations are not fixed on the boundary, then we obtain in the proof of Theorem 12 triangulations (20) such that the first $[\frac{n}{2}]$ triangulations have affine independent boundary $f$-vectors, and the last $[\frac{n+1}{2}]+1$ have the same boundary and have affine independent $f$-vectors. For the same reason as in the empty boundary case, we may further assume

$$\begin{aligned} &D(n)f(M; \Delta_i) - D(n)f(M; \Delta_{\delta_0}) \\ &= f(\partial M; \partial\Delta_i) - f(\partial M; \partial\Delta_{\delta_0}), \end{aligned} \quad 1 \leq i \leq [\tfrac{n}{2}] \tag{25}$$

integrally span a direct summand of $\mathbf{Z}^n$, and

$$f(M; \Delta_{\delta_j}) - f(M; \Delta_{\delta_0}), \quad 1 \leq j \leq [\tfrac{n+1}{2}] \tag{26}$$

integrally span a direct summand of $\mathbf{Z}^{n+1}$.

To go from Theorems 11 and 12 (which are of rational nature) to Theorems 2 and 3 (which are of integral nature), we make use of the following algebraic result.

**Lemma 14** *Suppose*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \tag{27}$$

*is a sequence of finitely generated abelian groups. Suppose*

1. *$B$ is torsionless;*

2. *$\beta\alpha = 0$;*

3. *$\alpha$ is injective with direct summand image;*

4. *$\mathrm{rank}A = \mathrm{rank}(\ker\beta)$.*

*Then the sequence* (27) *is exact.*

*Proof*: The second condition means that we only need to show $\ker\beta \subset \mathrm{im}\alpha$. The last condition means that (27) is rationally exact, i.e., if $x \in \ker\beta$, then for some integer $m \neq 0$ we have $mx \in \mathrm{im}\alpha$. By the third condition, we may write $B = \mathrm{im}\alpha \oplus D$. For $x \in \ker\beta$, write $x = (\alpha(y), z)$. Then $mx = (\alpha(my), mz) \in \mathrm{im}\alpha \subset \mathrm{im}\alpha \oplus D$ means that $mz = 0$. Since $m \neq 0$ and $B$ is torsionless, we see that $z = 0$. Consequently, $x \in \mathrm{im}\alpha$. We thus proved $\mathrm{im}\alpha = \ker\beta$ and the exactness of (27).

*Proof of Theorems 2 and 3*: Denote

$$\begin{aligned} A = \quad & \mathbf{Z}\text{-aspan} f(M; \text{condition}) - f(M; \Delta_0) \\ = \quad & \mathbf{Z}\text{-span}[f(M; \text{condition}) - f(M; \Delta_0)], \end{aligned} \tag{28}$$

where $\Delta_0$ is any specific choice of triangulations. Then Theorem 2 and 3 are equivalent to

$$A = \begin{cases} \ker(D(n), \chi) & \text{if } \partial M = \emptyset \\ \ker\chi & \text{if } \partial M \neq \emptyset \end{cases} \tag{29}$$

We already know that the $A$ is contained in the right. To prove equality, we construct similar integral linear span $A'$ from the differences of $f$-vectors of (18) or (20). Then $A' \subset A$. If we can show that $A'$ is equal to the right of (29), then we may conclude that $A = A'$ and is also equal to the right of (29).

In case $\partial M = \emptyset$ (or more generally, the triangulations restrict to a prescribed one on boundary), we take

$$A' = \mathbf{Z}\text{-span}\{f(M;\Delta_{\delta_i}) - f(M;\Delta_{\delta_0}), 1 \le i \le [\tfrac{n+1}{2}]\}$$

to be the integral span of (24). So $A'$ may be assumed to be a direct summand of $\mathbf{Z}^{n+1}$. Now we apply Lemma 14 to

$$0 \to A' \xrightarrow{\text{incl}} \mathbf{Z}^{n+1} \xrightarrow{(D(n),\chi)} \mathbf{Z}^{n+1}.$$

The first condition of Lemma 14 is trivially satisfied The second condition follows from $A' \subset A$. The direct summand property of (24) says that the third condition of Lemma 14 is satisfied. Theorem 11 implies that the fourth condition of Lemma 14 are satisfied. Thus we may conclude that $A' = \ker(D(n),\chi)$. This completes the proof in case the boundary is empty.

In case $\partial M \neq \emptyset$ and the restriction of the triangulations on the boundary is not fixed, we take

$$A' = \mathbf{Z}\text{-span}\left\{\begin{array}{ll} f(M;\Delta_i) - f(M;\Delta_{\delta_0}), & 1 \le i \le [\frac{n}{2}] \\ f(M;\Delta_{\delta_j}) - f(M;\Delta_{\delta_0}), & 1 \le j \le [\frac{n+1}{2}] \end{array}\right\}$$

to be the integral span of (20). We also consider the span

$$\begin{aligned} A'_{n-1} &= \mathbf{Z}\text{-span}\{f(\partial M;\partial\Delta_i) - f(\partial M;\partial\Delta_{\delta_0}), \quad 1 \le i \le [\tfrac{n}{2}]\} \\ &= D(n)\mathbf{Z}\text{-span}\{f(M;\Delta_i) - f(M;\Delta_{\delta_0}), \quad 1 \le i \le [\tfrac{n}{2}]\} \end{aligned}$$

of "boundary differences" (25), and the span

$$A'_n = \mathbf{Z}\text{-span}\{f(M;\Delta_{\delta_j}) - f(M;\Delta_{\delta_0}),\quad 1 \le j \le [\tfrac{n+1}{2}]\}$$

of "interior differences" (26). We may assume $A'_{n-1}$ and $A'_n$ are direct summands in respective spaces.

Then we consider the commutative diagram

$$\begin{array}{ccccccc} & 0 & & & & & \\ & \downarrow & & & & & \\ 0 \to & A'_n & \xrightarrow{\text{incl}} & \mathbf{Z}^{n+1} & \xrightarrow{(D(n),\chi)} & \mathbf{Z}^{n+1} & \\ & \downarrow & & \| & & \downarrow p_{n+1} & \\ 0 \to & A' & \xrightarrow{\text{incl}} & \mathbf{Z}^{n+1} & \xrightarrow{\chi} & \mathbf{Z} & \\ & \downarrow D(n) & & \downarrow D(n) & & \downarrow \imath_n & \\ 0 \to & A'_{n-1} & \xrightarrow{\text{incl}} & \mathbf{Z}^n & \xrightarrow{(D(n-1),\chi)} & \mathbf{Z}^n & \\ & \downarrow & & & & & \\ & 0 & & & & & \end{array} \tag{30}$$

where $p_{n+1}$ is the projection to the last coordinate, and

$$\imath_n(a) = \begin{cases} (0,0,\cdots,0,0) & n \text{ is even} \\ (0,0,\cdots,0,2a) & n \text{ is odd} \end{cases}.$$

The commutativity of the diagram (especially the lower right square) follows from (4) and (5).

Since $\Delta_{\delta_j}$ are assumed to have the same boundary triangulation, we have

$$D(n)[f(M;\Delta_{\delta_j}) - f(M;\Delta_{\delta_0})] = f(\partial M;\partial\Delta_{\delta_j}) - f(\partial M;\partial\Delta_{\delta_0}) = 0.$$

Therefore $D(n)A'_n = 0$. Moreover, $D(n)$ sends $f(M;\Delta_i) - f(M;\Delta_{\delta_0}) \in A'$ to the basis of $A'_{n-1}$. Thus we see that the column on the left of (30) is exact.

We have proved in the empty boundary case that the bottom row of (30) is exact. By the same argument we may prove that the top row of (30) is also exact. Also recall that $A' \subset A \subset \ker\chi$ at the middle row.

Now we are ready to prove that $\ker\chi \subset A'$, i.e., the middle row is exact. Suppose that $\chi(x) = 0$. Then the commutativity of the lower right square of (30) implies that $(D(n-1), \chi)D(n)x = \imath_n\chi(x) = 0$. By the exactness of the bottom row and the left column, we have $D(n)x = D(n)a$ for some $a \in A'$. Then from $\chi(A') = 0$ we get $\chi(x-a) = \chi(x) - \chi(a) = 0 - 0 = 0$. Combined with $(D(n), \chi)(x-a) = 0$, the exactness of the top row then implies that $x - a \in A'_n \subset A'$. Therefore we conclude that $x \in A'$. This completes the proof in case the boundary is not empty.

We remark that $A'_n$ and $A'_{n-1}$ are in fact independent of specific manifolds. Given any manifold $M$ without boundary, we have

$$\mathbf{Z}\text{-aspan} f(M; \text{condition}) = A'_n + f(M; \Delta_0),$$

and more generally

$$\mathbf{Z}\text{-aspan} f(M; \text{rel } T, \text{ condition}) = A'_n + f(M; \Delta_0),$$

where $\partial\Delta_0 = T$. The exactness of left column of (30) then means the sequence

$$0 \to \mathbf{Z}\text{-aspan} f(M; \text{rel } T) \xrightarrow{\text{incl}} \mathbf{Z}\text{-aspan} f(M) \xrightarrow{D(n)} \mathbf{Z}\text{-aspan} f(\partial M) \to 0$$

is affine exact.

# 7 Torsion Linear Conditions

The algebraic tool for dealing with the problem of torsion linear conditions is the following Lemma.

**Lemma 15** *Suppose*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

*is an exact sequence of abelian groups, such that the image of $\beta$ is a direct summand of $C$. Then for any linear function $\lambda : B \to \mathbf{Z}/k\mathbf{Z}$ such that $\lambda\alpha = 0$, there is a linear function $\mu : C \to \mathbf{Z}/k\mathbf{Z}$ with $\lambda = \mu\beta$.*

*Proof*: By exactness, the equality $\lambda\alpha = 0$ means precisely that $\lambda$ factors through the image of $\beta$. If the image of $\beta$ is further assumed to be a direct summand, then the factorization may extend to a map on $C$. This map is our $\mu$.

*Proof of Theorem 4*: A torsion condition is a linear function $\lambda : \mathbf{Z}^{n+1} \to \mathbf{Z}/k\mathbf{Z}$ that restricts to a constant modulo $k$ on $\mathbf{Z}$-aspan$f(M; \text{condition})$. This is equivalent to that $\lambda$ restricts to 0 modulo $k$ on $A$ of (28). It has been explained in the proof of Theorem 2 that the Theorem may be interpreted as the exactness of the sequence

$$0 \to A \xrightarrow{\text{incl}} \mathbf{Z}^{n+1} \xrightarrow{(D(n),\chi)} \mathbf{Z}^{n+1}$$

in case the triangulations are fixed on $\partial M$, or the exactness of the sequence

$$0 \to A \xrightarrow{\text{incl}} \mathbf{Z}^{n+1} \xrightarrow{\chi} \mathbf{Z}$$

in case $\partial M \neq \emptyset$ and the triangulations are not fixed on $\partial M$. If the condition of Lemma 15 may be verified for the two sequences, then the conclusion $\lambda = \mu\beta$ means precisely that $\lambda$ is a linear combination of the Dehn-Sommerville equations and the Euler equation in the first case, or a multiple of the Euler equation in the second case.

Therefore the proof of Theorem 4 is reduced to showing that the images of $(D(n), \chi)$ and $\chi$ are direct summands. The image of $\chi$ is the whole $\mathbf{Z}$ and so is trivially a direct summand.

To prove that the image of $(D(n), \chi)$ is a direct summand, we make use of the equality (5) and Theorem 7.

If $n$ is odd, we have from (4) and (5) that

$$\begin{pmatrix} D(n-1) & 0 \\ \chi & -2 \end{pmatrix} \begin{pmatrix} D(n) \\ \chi \end{pmatrix} = 0.$$

Conversely, suppose that $D(n-1)x = 0$ and $\chi(x) = 2a$. Then $\chi(x) = 0$ modulo 2, and by Theorem 7, $[x] \in H_{n-1}DS$ vanishes. This means that $x = D(n)y$ for some $y$. Then it follows from (5) that $2a = \chi(D(n)y) = 2\chi(y)$. Thus we have $(x, a) = (D(n), \chi)y$. So we proved that

$$\operatorname{im} \begin{pmatrix} D(n) \\ \chi \end{pmatrix} = \ker \begin{pmatrix} D(n-1) & 0 \\ \chi & -2 \end{pmatrix}.$$

Since the kernel of any linear map $\mathbf{Z}^{n+1} \to \mathbf{Z}^{n+1}$ is a direct summand, we see that the image of $(D(n), \chi)$ is a direct summand of $\mathbf{Z}^{n+1}$.

If $n$ is even, we have from (4) that

$$(D(n-1), 0) \begin{pmatrix} D(n) \\ \chi \end{pmatrix} = 0.$$

Conversely, suppose that $D(n-1)x = 0$ and $a$ is any integer. Then by Theorem 7, $[x] \in H_{n-1}DS = 0$. This means that $x = D(n)y$ for some $y$. Let

$$z = y + (a - \chi(y), 0, 0, \cdots).$$

Then it follows from the expression of the matrix $D(n)$ (the first column consists of zeros) that $D(n)z = D(n)y = x$ and $\chi(z) = \chi(y) + (a - \chi(y)) = a$. So we proved that

$$\mathrm{im}\begin{pmatrix} D(n) \\ \chi \end{pmatrix} = \ker(D(n-1), 0).$$

It again follows that the image of $(D(n), \chi)$ is a direct summand of $\mathbf{Z}^{n+1}$.

This completes the proof of Theorem 4.

# References

[Ba] Barnette, D.W.: Diagrams and Schlegel diagrams. In: Combinatorial Structure and their Applications (Proc. Calgary Internat. Conf., 1969). New York: Gordon and Beach 1970

[BB] Bayer, M. M., and Billera, L. J.: Generalized Dehn-Sommerville relations for polytopes, spheres, and Eulerian partially ordered sets. Invent. Math. **79**, 143-157 (1985)

[BL1] Bayer, M., Lee, C.: Combinatorial aspects of convex polytopes. In: Handbook of Convex Geometry, Gruber, P.M., Wills, J.M.(ed), 485-534. Elsevier Science 1993

[BL2] Billera, L. J., Lee, C.: A proof of the sufficiency of McMullen's conditions for $f$-vectors of simplicial convex polytopes, Jour. Combinatorial Theory (A) **31**, 237-255 (1981)

[Br] Brøndsted, A.: An Introduction to Convex Polytopes. Graduate Texts in Math. vol.**90**. Berlin-Heidelberg-New York: Springer-Verlag 1983

[G1] Grünbaum, B.: Convex Polytopes. Pure and Applied Math. vol.**16**. New York: Wiley-Interscience 1967

[G2] Grünbaum, B.: Polytopes, graphs, and complexes. Bull. Amer. Math. Soc. vol.**76**, 1131-1201 (1970)

[GS] Grünbaum, B., Sreedharan, V.P.: An enumeration of 4-polytopes with 8 vertices. Jour. Combinatorial Theory (A) **2**, 437-465 (1967)

[K] Klee, V.: A combinatorial analogue of Poincaré's duality theorem. Can. J. Math. **16**, 517-531 (1964)

[M] McMullen, P.: The number of faces of simplicial polytopes. Israel Jour. Math. **9**, 559-570 (1971)

[MS] McMullen, P. and Shephard, G. C.: Convex polytopes and upper bound conjecture. London Math. Soc. Lecture Note Ser. **3** London: Cambridge University Press 1971

[S1] Stanley, R. P.: On the number of faces of simplicial convex polytopes. Advances in Math. **35**, 236-238 (1980)

[S2] Stanley, R. P.: The number of faces of simplicial polytopes and spheres, in Discrete Geometry and Convexity (Goodman, J. E. et al, eds.), Ann. N. Y. Acad. Sci., vol.**440**, New York Academy of Sciences, New York, NY, 212-223 (1985)