f-Vectors of Polyhedra

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Abstract

The f-vector of a triangulation of a polyhedron X is the numbers of simplices at various dimensions. We prove that the affine span of f-vectors of X has dimension $\frac{n+s+1}{2}$, where n is the dimension of X, and s is the dimension of the part of X that is singular with respect to the local Euler characteristic.

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1 Statement of Results

Let X be a polyhedron of dimension n. Given a triangulation Δ of X, let $f_i(X;\Delta)$ be the number of simplices of dimension i in the triangulation. The *f-vector* of the triangulation is

$$f(X;\Delta) = (f_0, f_1, \cdots, f_n).$$

Kruskal and Katona [Ka, Kr] obtained necessary and sufficient conditions for a vector to be the f-vector of some simplicial complex. Jungerman and Ringel [JR, R] gave a complete characterization of the f-vectors of triangulations of compact surfaces. The f-vectors of simplicial polytopes have been extensively studied [BaL, Br, G1, G2, MS]. McMullen [M] conjectured the necessary and sufficient conditions for a vector to be the f-vector of some simplicial polytope. Billera and Lee [BiL] proved the sufficiency part of the conjecture, and Stanley [St] proved the necessity part.

Among the necessary conditions for f-vectors are the Euler characteristic equation and certain linear conditions called the Dehn-Sommerville equations. Such equations were first established for simplicial polytopes [So] and were proved to be the only linear conditions on the f-vectors [Br, G1]. Klee [Kl] generalized the Dehn-Sommerville equations to Euler manifolds (simplicial complexes with same local Euler characteristic property as combinatorial manifolds without boundary). Chen and the author [CY1,CY2] further generalized the equations to Euler manifolds with boundary and obtained all the linear conditions for f-vectors of triangulations of such spaces.

The main result of this paper is the characterization of all (rational) linear conditions on f-vectors of all triangulations of a (fixed) compact polyhedron. In particular, we obtain the dimension of the affine space spanned by these f-vectors.

To state the result, we need the following definition.

Definition 1 Let X be an n-dimensional polyhedron. A point $x \in X$ is called an *(Eulerian) regular point* if $\chi(\text{lk}(x,X)) = \chi(S^{n-1}) = 1-(-1)^n$. A point that is not Eulerian regular is called an *(Eulerian) singular point*.

$\text{lk}(x,X)$ is the link of the point x in X. It depends only on the local PL-structure of X at x. It is compact and unique up to PL-homeomorphisms. More details on links of points (and links of simplices introduced later on) may be found in Rourke and Sanderson's book on PL-topology [RS].

We will show (Lemma 3) that for any given triangulation of X, the Eulerian singular part is the union of the interiors of some simplices. The dimension s of the Eulerian singular part is then defined as the biggest dimension of these simplices. We will further show (Lemma 4) that n and s have different parity. The dimension of the affine span of the f-vectors is then given by the following theorem.

Theorem 1 *Let X be a polyhedron of dimension n. Let s be the dimension of the Eulerian singular part. Then the dimension of the affine span of the f-vectors of all triangulations of X is $\frac{n+s+1}{2}$.*

If X has no singular parts, then we should take $s=-1$ in case n is even, and $s=0$ in case n is odd. The conclusion of the theorem is then consistent with the classical result on f-vectors for simplicial polytopes [Br, G1] and the extension to f-vectors for manifolds without boundary [CY1,CY2].

If X is a manifold with nonempty boundary ∂X, then ∂X is exactly the Eulerian singular part of X. Therefore $s = n - 1$, and the conclusion of the theorem is consistent with the result of Chen and the author in this case [CY1].

The theorem is the consequence of the following explicit characterization of the affine span.

Theorem 2 *Let X be a polyhedron of dimension n. Let s be the dimension of the Eulerian singular part. Then the (rational) affine span of the f-vectors of all triangulations of X consists of vectors $(f_0, f_1, \cdots, f_n)$ satisfying the last $(n - s - 1)$ Dehn-Sommerville equations*

$$(1 - (-1)^{n-i})f_i + \sum_{j>i}(-1)^{n-j-1}\binom{j+1}{i+1} f_j = 0, \quad s < i < n,$$

and the Euler characteristic equation

$$\sum_{j=0}^{n}(-1)^j f_j = \chi(X).$$

It is well-known that the last $(n - s - 1)$ Dehn-Sommerville equations have rank $\frac{n-s-1}{2}$, and the Euler equation is linearly independent of these Dehn-Sommerville equations. Therefore the dimension of the affine space is $(n+1) - \frac{n-s-1}{2} - 1 = \frac{n+s+1}{2}$.

It should be pointed out that the linear relations satisfied by the f-vectors are the only ones with rational coefficients. It is quite possible to have further torsion relations (such as $\lambda(f) = 0 \bmod p$) with integer coefficients. In [CY2], such a possibility is extensively studied for the case that the local Euler characteristic $\chi(\mathrm{lk}(x, X))$ takes only two values. It is not very difficult to extend the discussion to the general case and explicitly write down all the torsion relations. The result depends on the stratification structure given by the local Euler characteristic.

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2 Dehn-Sommerville Equations for Polyhedra

We classify points of X according to their local Euler characteristics

$$X_k = \{x \in X : \chi(\mathrm{lk}(x, X)) = k\}.$$

Thus $X_{1-(-1)^n}$ is the Eulerian regular part of X, and $\cup_{k \neq 1-(-1)^n} X_k$ is the Eulerian singular part.

If all points are regular, then X is called an *Euler manifold without boundary*. The notion was first defined in [Kl] for simplicial complexes and was modified in the present form for polyhedra in [CY2]. In fact, the concept was further extended in [CY2] to include manifolds with boundary (and more generally, 2-strata spaces). If there is only one type of singular point, i.e., $X_k = \emptyset$ for all k except $1-(-1)^n$ and k_0, then X is called an *Euler manifold with boundary* $\partial X = X_{k_0}$.

Let Δ be a triangulation of X. Then for any simplex σ of Δ, we have the *simplicial link of* σ *in* Δ

$$\mathrm{lk}(\sigma, \Delta) = \{\tau \in \Delta : \tau \cap \sigma = \emptyset, \tau \text{ and } \sigma \text{ span a simplex } \sigma * \tau \text{ of } \Delta\}.$$

The PL-homeomorphism type of its geometrical realization $\mathrm{lk}(\sigma, X)$ does not depend on the triangulation and is called the *link of* σ *in* X. Denote by $\dot{\sigma}$ the interior of σ. Then for $x \in \dot{\sigma}$, we have the following relation

$$\mathrm{lk}(x, X) = \partial\sigma * \mathrm{lk}(\sigma, X). \tag{1}$$

Lemma 3

$$X_k = \cup\{\dot{\sigma} : \chi(\mathrm{lk}(\sigma, X)) = (-1)^{\dim\sigma}(k - 1 + (-1)^{\dim\sigma})\}.$$

Proof: Suppose that $x \in \dot{\sigma}$, $\dim\sigma = i$. Then it follows from (1) that

$$\begin{aligned} \chi(\mathrm{lk}(x, X)) = &\ \chi(S^{i-1}) + \chi(\mathrm{lk}(\sigma, X)) - \chi(S^{i-1})\chi(\mathrm{lk}(\sigma, X)) \\ = &\ 1 - (-1)^i + (-1)^i\chi(\mathrm{lk}(\sigma, X)). \end{aligned}$$

This equality is equivalent to

$$\chi(\mathrm{lk}(\sigma, X)) = (-1)^i(\chi(\mathrm{lk}(x, X)) - 1 + (-1)^i).$$

The Lemma is then proved.

To prove that n and s have different parity, we need the following facts from [CY2]:

1. If X is an Euler manifold with boundary ∂X, then $\mathrm{lk}(\sigma, X)$ is an Euler manifold with boundary $\mathrm{lk}(\sigma, \partial X)$. In particular, $\mathrm{lk}(\sigma, X)$ is an Euler manifold without boundary unless σ is the proper face of some simplex in ∂X. Moreover, $\dim \mathrm{lk}(\sigma, X)$ and $n - \dim\sigma$ have different parity;

2. If X is an Euler manifold of odd dimension and without boundary, then $\chi(X) = 0$.

Lemma 4 *The dimension* s *of the singular part has different parity from the dimension* n *of the whole polyhedron.*

Proof: Let σ be a simplex of dimension s such that $\dot{\sigma} \subset X_k$, $k \neq 1 - (-1)^n$. Because σ has the highest singular dimension, the interior of any simplex with σ as a proper face consists of regular points. Let U be the union of $\dot{\sigma}$ with the interiors of these simplices. Then U is an open neighborhood of $\dot{\sigma}$ such that $\dot{\sigma}$ is closed in U and $U - \dot{\sigma} \subset X_{1-(-1)^n}$. In other words, U is an Euler manifold with boundary $\dot{\sigma}$. Take any locally finite triangulation T of $(U, \dot{\sigma})$ and a simplex $\tau \subset \dot{\sigma}$ of T of dimension s. By the first property of Euler manifolds and the fact that $\dim\tau = \dim\sigma$, $\mathrm{lk}(\tau, U)$

is an Euler manifold without boundary. Moreover, its dimension has different parity from $n-s$.

Therefore if $n-s$ is even, then $\mathrm{lk}(\tau, U)$ is an odd dimensional Euler manifold. By the second property about Euler manifolds, we have $\chi(\mathrm{lk}(\tau, U)) = 0$. Therefore if $x \in \dot{\tau}$, then by (1) we have $\chi(\mathrm{lk}(x, X)) = \chi(\mathrm{lk}(x, U)) = \chi(\partial\tau * \mathrm{lk}(\tau, U)) = \chi(\partial\tau) = \chi(S^{s-1}) = \chi(S^{n-1})$. This contradicts $x \in \dot{\tau} \subset \dot{\sigma} \subset X_k$, $k \neq 1-(-1)^n$.

This completes the proof of Lemma 4.

Denote by $f_i(X_k; \Delta)$ the number of simplices σ of Δ such that $\dot{\sigma} \subset X_k$. Then $f_i(X; \Delta) = \sum_k f_i(X_k; \Delta)$, and we have

$$\begin{aligned} \textstyle\sum_{j>i}(-1)^{j-i-1}\binom{j+1}{i+1} f_j(X;\Delta) &= \textstyle\sum_{\dim\sigma=i}\chi(\mathrm{lk}(\sigma,\Delta)) \\ &= \textstyle\sum_k(-1)^i(k-1+(-1)^i)f_i(X_k;\Delta), \end{aligned} \tag{2}$$

where the first equality was essentially due to Klee [Kl] (a reformulation of Klee's argument may be found in [CY1]), the second equality is due to Lemma 3. Thus we get the following Dehn-Sommerville equation

$$\begin{aligned} & (1-(-1)^{n-i})f_i(X;\Delta) + \textstyle\sum_{j>i}(-1)^{n-j-1}\binom{j+1}{i+1} f_j(X;\Delta) \\ = & \textstyle\sum_k(1-(-1)^{n-i})f_i(X_k;\Delta) + \sum_k(-1)^n(k-1+(-1)^i)f_i(X_k;\Delta) \\ = & \textstyle\sum_k(1-(-1)^n+(-1)^n k)f_i(X_k;\Delta). \end{aligned}$$

The coefficients of $f_*(X_k; \Delta)$ on the left side form the *Dehn-Sommerville matrix* $D(n)$ introduced in [CY1]. We observe that the coefficient of $f_i(X_k; \Delta)$ on the right side vanishes if and only if $k = 1-(-1)^n = \chi(S^{n-1})$. Therefore we conclude the following.

Lemma 5

$$D(n)f(X;\Delta) = \sum_{k\neq 1-(-1)^n}(1-(-1)^n+(-1)^n k)f(X_k;\Delta). \tag{3}$$

3 Proof of Main Result

By Lemma 4, we see that n and $s+1$ have the same parity. Then we may conclude from the explicit expression of $D(n)$ that

$$D(n) = \begin{pmatrix} D(s+1) & * \\ 0 & E(n,s+1) \end{pmatrix}. \tag{4}$$

The statement of the theorem 2 is then the equality

$$\text{affine span} f(X) = \{v \in \mathbf{Q}^{n+1} : (0, E(n,s+1))v = 0, \chi(v) = \chi(X)\}. \tag{5}$$

Given any triangulation Δ of X, the right side of the Dehn-Sommerville equation (3) is a combination of f-vectors of spaces of dimension $\leq s$. Therefore the last $n-s-1$ coordinates $(0, E(n,s+1))f(X;\Delta)$ of the n-dimensional vector $D(n)f(X;\Delta)$ must vanish. This shows that the left side of (5) is contained in the right side.

To show the two sides to be the same, we only need to find $\frac{n+s+1}{2}+1$ triangulations of X with affinely independent f-vectors. Here we make use of the existence of triangulations δ_i^p of D^p, $0 \leq i \leq \left[\frac{p+1}{2}\right]$, with the following properties

1. $\partial\delta_i^p = \delta_i^p|_{S^{p-1}} = \partial\Delta^p$, the boundary of the standard p-simplex;

2. The f-vectors $f(D^p;\delta_i^p)$ are affinely independent.

Such triangulations may be obtained by deleting one p-dimensional simplex from some triangulations of S^p (the boundaries of $(p+1)$-dimensional cyclic polytopes [Br, G1], for example).

We start by fixing an arbitrary triangulation Δ of X. Let $\dot{\sigma} \subset X_k$, $k \neq 1-(-1)^n$, be the interior of an s-dimensional simplex in the singular part of X. Then we replace the subcomplex $\sigma * \mathrm{lk}(\sigma,\Delta)$ of Δ by the complex $\delta_i^s * \mathrm{lk}(\sigma,\Delta)$. Since $\sigma * \mathrm{lk}(\sigma,\Delta)$ is glued to the rest of Δ along $\partial\sigma * \mathrm{lk}(\sigma,\Delta) = \partial\delta_i^s * \mathrm{lk}(\sigma,\Delta)$ (the equality is by the first property of δ_i^s), the replacement is still a triangulation of X. We denote the triangulation by Δ_i.

For any simplex $\tau \in \mathrm{lk}(\sigma,\Delta)$, the simplex $\sigma * \tau$ is replaced by $\delta_i^s * \tau$. Since σ has the highest singular dimension, the interior of $\sigma * \tau$ consists of regular points. It then follows from $\partial\delta_i^s = \partial\sigma$ that the subdivision does not affect the singular part of X except the interior of σ. Consequently,

$$f(X_l;\Delta_i) = f(X_l;\Delta), \quad l \neq k, 1-(-1)^n \tag{6}$$

and

$$f(X_k;\Delta_i) = f(X_k;\Delta) + f(D^s;\delta_i^s) - f(S^{s-1};\partial\Delta^s), \quad l = k. \tag{7}$$

By Dehn-Sommerville equation (3), we see that (6) and (7) imply

$$\begin{array}{rl} & D(n)f(X;\Delta_i) - D(n)f(X;\Delta_0) \\ = & (1-(-1)^n + (-1)^n k)[f(D^s;\delta_i^s) - f(D^s;\delta_0^s)], \end{array} \quad 1 \leq i \leq [\tfrac{s+1}{2}] \tag{8}$$

These are linearly independent by the second property of δ_i^s.

The similar replacement may be carried out in the regular part of X. Let $\dot{\theta} \subset X_{1-(-1)^n}$ be the interior of an n-dimensional simplex of Δ_0, the zeroth of $\left[\frac{s+3}{2}\right]$ triangulations we just constructed. Then we may replace θ with the complex δ_j^n to obtain $\Delta_{\delta_{0,j}}$, $0 \leq j \leq \left[\frac{n+1}{2}\right]$. Since $\partial\delta_j^n = \partial\theta$, nothing is changed on simplices other than the interior $\dot{\theta}$. Therefore, we have

$$f(X_l;\Delta_{0,j}) = f(X_l;\Delta_0), \quad l \neq 1-(-1)^n, \tag{9}$$

and

$$f(X;\Delta_{0,j}) = f(X;\Delta_0) + f(D^n;\delta_j^n) - f(S^{n-1};\partial\Delta^n). \tag{10}$$

By the second property of δ_j^n, (10) implies that the vectors

$$f(X;\Delta_{0,j}) - f(X;\Delta_{0,0}) = f(D^n;\delta_j^n) - f(D^n;\delta_0^n), \quad 1 \leq j \leq [\tfrac{n+1}{2}] \tag{11}$$

are linearly independent.

Now we claim that the f-vectors of the triangulations

$$\Delta_1, \Delta_2, \cdots, \Delta_{\left[\frac{s+1}{2}\right]}; \quad \Delta_{0,0}, \Delta_{0,1}, \cdots \Delta_{0,\left[\frac{n+1}{2}\right]} \tag{12}$$

are affinely independent. Consider the linear relation

$$\sum_{i=1}^{\left[\frac{s+1}{2}\right]} a_i(f(X;\Delta_i)-f(X;\Delta_{0,0}))+\sum_{j=1}^{\left[\frac{n+1}{2}\right]} b_j(f(X;\Delta_{0,j})-f(X;\Delta_{0,0}))=0. \tag{13}$$

The Dehn-Sommerville equations (3) and the equation (9) imply that

$$D(n)f(X;\Delta_{0,j})=D(n)f(X;\Delta_0).$$

Therefore applying $D(n)$ to (13) gives rise to

$$\sum_{i=1}^{\left[\frac{s+1}{2}\right]} a_i(D(n)f(X;\Delta_i)-D(n)f(X;\Delta_0))=0. \tag{14}$$

Then the linear independence of (8) implies that $a_i = 0$ in (14). Moreover, (13) becomes

$$\sum_{j=1}^{\left[\frac{n+1}{2}\right]} b_j(f(X;\Delta_{0,j})-f(X;\Delta_{0,0}))=0.$$

Because (11) are linearly independent, we also conclude that $b_j = 0$.

The existence of triangulations (12) with affinely independent f-vectors means that the dimension of the left side of (5) is at least $\left[\frac{s+1}{2}\right]+\left[\frac{n+1}{2}\right]=\frac{n+s+1}{2}$, which is equal to the dimension of the right side. Consequently both sides of (5) are equal. This completes the proof of Theorem 2.

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