Equivariant Periodicity in Surgery for Actions of Some Nonabelian Groups

Min Yan*
Department of Mathematics
Hong Kong University of Science and Technology
Clear Water Bay, Hong Kong

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Abstract

The isovariant homotopy classification was proved to have equivariant periodicity $S_G(M, \mathrm{rel}\ \partial) \cong S_G(M \times DV, \mathrm{rel}\ \partial)$ for the following cases: V is the four fold permutation representation of an odd order group [Y], or V is twice of any complex representation of a compact abelian group [WY]. In this paper, the equivariant periodicity is proved for twice of the natural complex representation of $SU(2)$, and twice of any complex representation of $O(2)$, providing further evidence that the equivariant periodicity should be true for twice of any complex representation of any compact Lie group.

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Let $S(M, \mathrm{rel}\ \partial M)$ be the homeomorphism classes of topological manifolds that are homotopy equivalent to M and restrict to homeomorphisms on the boundary. Siebenmann proved that the homotopy classification of topological manifolds has the four fold periodicity (Appendix C of [KS]):

$$S(M, \mathrm{rel}\ \partial M) \cong S(M \times D^4, \mathrm{rel}\ \partial(M \times D^4)).$$

As pointed out by Nicas, this is in fact not quite correct in the context of manifolds [N]. The deviation may be a copy of $\mathbf{Z}$. On the other hand, this is indeed true if the consideration is enlarged to ANR-homology manifolds [BFMW]. Our purpose is to extend the periodicity to the isovariant (i.e., preserving the isotropy groups) homotopy classification of equivariant topological manifolds

$$S_G(M, \mathrm{rel}\ \partial M) \cong S_G(M \times DV, \mathrm{rel}\ \partial(M \times DV)),$$

where DV is the unit ball of some unitary G-representation V. Because of the fact that G may acts nontrivilly on DV , such kind of generalization is theoretically and practically quite useful (see [WY] for a more detailed discussion). Again the exact isomorphism is true only in the context of equivariant ANR-homology manifolds. However, the deviation in the context of equivariant topological manifolds is several copies of $\mathbf{Z}$ and is well understood. Therefore we will not be concerned with the deviation and will pretend that the exact statement is true for topological manifolds.

Under some mild combinatorial and small gap conditions, the equivariant periodicity has been proved for the following cases:

1. G is a finite group. $V = \mathbf{R}S \otimes \mathbf{R}^4$ is a permutation representation, where S is a finite G-set such that all its orbits are of odd order [Y].

2. G is a compact abelian group, including the torus group. $V = W \oplus W$ is twice of some complex representation W [WY].

Moreover, it was suggested in [WY] that the equivariant periodicity should be true for twice of a complex representation of any (abelian or nonabelian) compact Lie group. This paper provides some evidence in this direction. The main result is the following theorem.

Theorem 0.1 *Suppose that M is a homotopically stratified G-manifold with codimension ≥ 3 gap. Suppose that V is a G-representation such that M and $M \times V$ have the same isotropy everywhere. Then there is a natural isomorphism*

$$S_G(M, \mathrm{rel}\ \partial M) \cong S_G(M \times DV, \mathrm{rel}\ \partial(M \times DV))$$

for the following cases:

(1) $V = \mathbf{C}^2 \oplus \mathbf{C}^2 = \mathbf{R}^8$ *is twice the natural representation of* $G = SU(2)$ *on* $\mathbf{C}^2$*;*

(2) $V = \mathbf{H} \otimes_{\mathbf{R}} \mathbf{R}^2 = \mathbf{R}^8$ *is the quaternionization of the natural representation of* $G = O(2)$ *on* $\mathbf{R}^2$*.*

The result remains true for the representations of G' induced from a group homomorphism $G' \to G$.

The codimension ≥ 3 gap condition is the following: If $H \subset K$ are subgroups of G, and $M^H_\alpha \supset M^K_\beta$ are two connected components of fixed point subsets. Then either $M^H_\alpha = M^K_\beta$ or $\dim M^H_\alpha \geq \dim M^K_\beta + 3$. The condition will make sure that the isovariant and equivariant connected fixed point components are in one-to-one correspondence, and the corresponding components have isomorphic fundamental groups. The assumption is useful because the surgery and K-obstructions are basically determined by such combinatorial data. Moreover, such gap condition is often satisfied in applications.

The condition that M and $M \times V$ have the same isotropy everywhere was introduced in [Y] and means the following: For any $x \in M$, there is an equivariant neighborhood U of x, such that $\mathrm{iso}(U) = \mathrm{iso}(U \times V)$, where "iso" means the collection of isotropy groups. The condition makes sure that the isovariant fixed point components of M and $M \times V$ are in one-to-one correspondence, a naturally necessary combinatorial condition for the structures on M and $M \times DV$ to be equivalent. Moreover, if M has codimension ≥ 3 gap, then the same isotropy everywhere condition implies that $M \times V$ also has codimension ≥ 3 gap.

The statement in the theorem about representations induced from group homomorphisms may be applied to subgroups. For example, we get periodicity for the dihedral groups as subgroups of $O(2)$.

From the representation theory, any complex representation of $O(2)$ is induced either from a representation of $\mathbf{Z}_2$ via the group homomorphism $\det : O(2) \to \{\pm 1\}$, or from the complexification of the natural $O(2)$-representation on $\mathbf{R}^2$ via a homomorphism $O(2) \to O(2)$. In fact, this is also true for the dihedral subgroups of $O(2)$. Therefore by repeatedly making use of the periodicity for the abelian group $\{\pm 1\}$ and the above theorem, we may conclude the following:

Corollary 0.2 *The equivariant periodicity holds for twice of any complex representation of $O(2)$ or D_{2n}.*

The proof of the theorem is similar to the one in [WY]. The key is to construct a periodicity space that contains the representation, so that away from the representation, we have a stratified $\pi - \pi$ structure. The Wall's $\pi - \pi$ theorem [Wa] can be generalized to show that the surgery obstructions always vanish for the product of the such $\pi - \pi$ structure with any equivariant manifold. Moreover, the Tate cohomology of the K-theoretical obstructions also vanish for such products. By making use of Weinberger's stratified surgery theory [We], we are then able to prove the periodicity.

This paper is organized as follows: In the first part, the key properties that enable us to prove the periodicity in [WY][Y] are analyzed. This motivates the definition 1.1 of stratified isovariant $\pi - \pi$ structures and the definition 1.8 of equivariant periodicity spaces. Some useful properties of these objects are also proved. In the second part, the periodicity spaces for actions by $SU(2)$ and $O(2)$ are constructed. The $SU(2)$-periodicity space is a simple straightforward generalization of the S^1-periodicity space in [WY]. The construction of $O(2)$-periodicity space requires much more effort. In the third part, the general periodicity theorem 3.1 is proved. Therefore if one can find periodicity spaces (according to the definition 1.8) for other group actions, then one would obtain corresponding periodicity statement.

The author would like to point out that the definition 1.8 is very likely to be superseded by better ones. I hope that with more advanced machinery, one may come up with a definition of periodicity spaces that enables us to prove the following conjecture posed in [WY]: Twice of any complex representation of any compact Lie group is a periodicity representation.

1 Definition of Periodicity Spaces

The periodicity in the classical (nonequivariant) surgery theory was induced from the following isomorphisms

$$L(X) \overset{\times \mathbf{CP}^2}{\cong} L(X \times \mathbf{CP}^2) \overset{\text{incl}}{\cong} L(X \times D^4, \text{rel } S^3). \tag{1}$$

The reason for $\times\mathbf{CP}^2$ to be isomorphic is that $\mathbf{CP}^2$ is a closed manifold of signature 1. The reason for the inclusion to be isomorphic is that $\mathbf{CP}^2$ is connected and simply connected. Therefore if we replace $\mathbf{CP}^2$ by any closed, connected, simply connected manifold P^{4p} of signature 1, and replace (D^4, S^3) by (D^{4p}, S^{4p-1}), then (1) is still isomorphic. Such a manifold P is a *periodicity manifold* for the classical surgery theory.

The *equivariant periodicity manifolds* for the equivariant surgery theory were introduced in [DS] and [Y] as a straightforward generalization of the classical periodicity manifolds. A G-manifold P is a G-periodicity manifold if for any subgroup H of G, P^H is closed, connected, simply connected, and has equivariant $\pi_0 WH$-signature 1. Here the signature condition means that the $\pi_0 WH$-invariant intersection form of P^H at the middle dimension is isomorphic to the form

$$\text{multiplication} : \mathbf{Z} \otimes \mathbf{Z} \to \mathbf{Z}, \quad \text{trivial group action}$$

up to adding equivariant hyperbolic symmetric forms.

Unfortunately, such equivariant periodicity manifolds do not arise easily. The candidate considered in [Y] is the following

$$P = \times^S \mathbf{CP}^2 = \mathbf{CP}^2 \times \mathbf{CP}^2 \times \cdots \times \mathbf{CP}^2 \quad (S \text{ copies}),$$

where S is a finite G-set, and G acts on P through permutation. It was then proved (theorem 3.16 of [Y]) that if the orbits in S are all of odd orders, then P is a periodicity manifold. Dovermann and Schultz considered the special case $S = G$ for a finite group G acting by left multiplication. In addition to the fact that P is a periodic manifold for odd order G, they also showed that P is not periodic for even order G (theorem 3.7(ii) of [DS], even for the general case that $\mathbf{CP}^2$ is replaced by any classical periodicity manifold).

It should also be noted that Browder suggested the construction of $P = \times^G \mathbf{CP}^2$ with the permutation action by a finite group G long time ago (1976 AMS Summer Symposium on Algebraic and Geometric Topology at Stanford). The purpose was to increase the dimensional gap between fixed point components of various subgroups to a big one by crossing with (maybe several copies of) P. The periodicity property of P implies that the process preserves much interesting geometric information, while the big gap enables people to apply many geometric machineries. The purpose is well served for odd order groups and will evidently not be achieved for even order ones. In fact, Dovermann and Schultz proved that one may never obtain big gaps by crossing with $\mathbf{Z}_2$-periodicity manifolds (theorem 3.7(i) of [DS]).

It appears that there is no useful periodicity for actions of even order groups by using only periodicity *manifolds*. However, the stratified approach to group actions motivates the construction in [WY] of a periodicity *space* for actions of the circle group S^1 and all cyclic groups. The resulting periodicity may be further extended to actions of all compact abelian groups. Moreover, such periodicity often increases the gap to a big one, for actions by abelian groups of odd as well as even orders.

The S^1-periodicity space considered in [WY] is

$$P = \mathbf{CP}^2 \cup D^3, \tag{2}$$

where D^3 is attached to $\mathbf{CP}^2$ via the identification $S^2 = \mathbf{CP}^1$. The action of S^1 is trivial on D^3, and by the formula

$$\lambda[z_1, z_2, z_3] = [\lambda z_1, \lambda z_2, z_3] \tag{3}$$

on $\mathbf{CP}^2$.

By $\mathbf{CP}^2 = D^4 \cup_{S^3 \nearrow} S^2$ (the map $S^3 \to S^2$ is the Hopf bundle map), P is an S^1-stratified space with three strata

$$P = D^4 \cup_{S^3 \nearrow} S^2_{\nwarrow S^2} \cup D^3. \tag{4}$$

S^1 acts on $D^4 \subset \mathbf{C}^2$ (the first two coordinates of (3)) by complex multiplication, and acts trivially on D^3 and S^2.

The following two properties enable us to prove the S^1-periodicity:

1. The signature of $\mathbf{CP}^2$ is 1, and the euler characteristic is 3, an odd number;

2. D^3 is a manifold with boundary S^2. Both are fixed by S^1 , and have isomorphic fundamental groups (the trivial group).

Strictly speaking, we should consider the $\pi_0 S^1$-equivariant signature and euler characteristic. However, these are the same as the nonequivariant one because $\pi_0 S^1$ is trivial.

The S^1-periodicity is similarly obtained by first considering (compare with (1)):

$$L_{S^1}(X) \overset{\times P}{\to} L_{S^1}(X \times P) \overset{\text{incl}}{\leftarrow} L_{S^1}(X \times D^4, \text{rel } S^3), \tag{5}$$

where the S^1-stratification on $X \times P$ is induced from that of P. The inclusion map fits into a long exact sequence in which the third term is $L_{S^1}(X \times (D^3, S^2))$. Because of the second property of P, each isovariant fixed point component of $X \times (D^3, S^2)$ is a $\pi - \pi$ pair. By the equivariant version of Wall's $\pi - \pi$ theorem [BQ], $L_{S^1}(X \times (D^3, S^2))$ vanishes. Consequently, the inclusion is an isomorphism. It is a little more complicated to show that $\times P$ is isomorphic. We simply point out that this essentially is a consequence of the first property of P. The fact about the euler characteristic is used in studying the map $\times P$ in the "destablization" stage of the homotopically stratified surgery theory.

It is quite conceivable that periodicity spaces may be constructed for actions of other compact Lie groups. In order to clarify the structure of such spaces, we first elaborate on the second property of (2) (4).

Let G be a compact Lie group. A G-stratified space X is generally indexed by a partially ordered set A. The closed strata of X will be denoted by X_α, while the (pure) open ones by X^α. X^α is a G-space.

X/G is a stratified space doubly indexed by A and the collection $\mathrm{Iso}(X)$ of isotropy subgroups of X. We say X is *homotopically G-stratified* if X/G is a homotopically stratified space.

Let B be a partially ordered set. Let ∂B be a copy of B, and $\beta \in B$ correspond to $\partial\beta \in \partial B$. Then the double $2B = B \coprod \partial B$ is partially ordered by keeping the original partial orders in B and ∂B, and adding the extra orders $\partial\beta < \beta$.

Definition 1.1 A homotopically G-stratified space X is an *isovariant $\pi - \pi$ structure* if X is indexed by the double $2B$ of some partially ordered set B, such that for any $\beta \in B$, $X^\beta \cup X^{\partial\beta}$ is a manifold with boundary $X^{\partial\beta}$, and the pair $(X^\beta \cup X^{\partial\beta}, X^{\partial\beta})$ satisfies the *isovariant $\pi - \pi$ condition*: the connected components of X^β and $X^{\partial\beta}$ are in one-to-one correspondence; and the corresponding ones have isomorphic fundamental groups.

The classical $\pi - \pi$ theorem has the following straightforward elaboration.

Proposition 1.2 *Suppose a homotopically G-stratified space X is an isovariant $\pi - \pi$ structure. Then $L_G(X)$ vanishes.*

Proof: Let β be a minimal element of B. Then $L_G(X)$ fits into a long exact sequence in which the other two terms are $L_G(X^\beta \cup X^{\partial\beta}, X^{\partial\beta})$ and $L_G(X - (X^\beta \cup X^{\partial\beta})) = L_G(X, \mathrm{rel}\ X^\beta \cup X^{\partial\beta})$. Since $(X^\beta \cup X^{\partial\beta}, X^{\partial\beta})$ satisfies the isovariant $\pi - \pi$ condition, $L_G(X^\beta \cup X^{\partial\beta}, X^{\partial\beta})$ vanishes by the equivariant $\pi - \pi$ theorem. Since $X - (X^\beta \cup X^{\partial\beta})$ is also an isovariant $\pi - \pi$ structure with smaller index set $2(B - \{\beta\})$, its surgery obstructions may be assumed to vanish by induction. Then we may conclude that $L_G(X)$ vanishes.

Remark 1.3 We will need the similar proposition for the other obstruction theories. By inspecting the proof of the proposition, we see that the following are the sufficient conditions for the proposition 1.2 to be valid for a stratified obstruction theory Λ:

1. (classical $\pi - \pi$ theorem) For an unstratified manifold pair $(X, \partial X)$ with X, ∂X connected, and $\pi_1 X = \pi_1 \partial X$, the obstruction $\Lambda(X, \partial X)$ is trivial;

2. (decomposition along singularities) For a stratified space X and a closed union Y of strata of X, $\Lambda(X)$ fits into a long exact sequence in which the other two terms are $\Lambda(Y)$ and $\Lambda(X - Y) = \Lambda(X, \mathrm{rel}\ Y)$.

The (nonstratified) equivariant $\pi - \pi$ theorem used in the proof of the proposition 1.2 is a consequence of the two conditions.

Because we will take product with periodicity spaces, we need to know the behavior of isovariant $\pi - \pi$ structures with respect to products.

Proposition 1.4 *Suppose X and Y are homotopically G-stratified spaces. If X is an isovariant $\pi - \pi$ structure, then $X \times Y$ is also an isovariant $\pi - \pi$ structure.*

Proof: Suppose the $\pi-\pi$ structure of X is indexed by $2B$, and Y is indexed by C. Then $X\times Y$ has a stratification indexed by $2(B\times C)$ with strata pairs $(X\times Y)^{(\beta,\gamma)}=X^\beta\times Y^\gamma$ and $(X\times Y)^{\partial(\beta,\gamma)}=X^{\partial\beta}\times Y^\gamma$. However, it is not immediately clear that the pair satisfies the isovariant $\pi-\pi$ condition.

The problem is that the isovariant components behave badly with respect to products ($X_H=\{x\in X:G_x=H\}$):

$$(X\times Y)_H=\cup_{H_1\cap H_2=H}X_{H_1}\times Y_{H_2}. \tag{6}$$

However, we may view the above expression as a stratification of $(X\times Y)_H$ with pure strata $X_{H_1}\times Y_{H_2}$. Therefore to complete the proof of the proposition, it suffices to show the following result and then apply it to the inclusion $X^{\partial\beta}\times Y^\gamma\subset(X^\beta\cup X^{\partial\beta})\times Y^\gamma$.

Proposition 1.5 *Suppose that $f:X\to Y$ is a homotopically transverse map between homotopically stratified spaces. If f is a one-to-one correspondence between the components of pure strata, and induces surjections (or isomorphisms) between fundamental groups of the corresponding components, then $f_*:\pi_1X\to\pi_1Y$ is surjective (or isomorphic).*

Proof: We may assume that $X=\bar{X}\cup E_X\times I\cup X_0$, where $\bar{X}$ is a top stratum of X, X_0 is a closed union of all strata except $\bar{X}$, and $E_X\to X_0$ is a stratified system of fibrations.

We may further assume that Y and f have similar decompositions, such that $f_E:E_X\to E_Y$ is a map of stratified systems of fibrations over the homotopically stratified map f_0.

By Vam-Kampen theorem, we have pushouts of fundamental groups and maps between them

$$\begin{array}{ccc} \pi_1E_X & \to & \pi_1X_0 \\ \downarrow & & \downarrow \\ \pi_1\bar{X} & \to & \pi_1X \end{array} \Longrightarrow \begin{array}{ccc} \pi_1E_Y & \to & \pi_1Y_0 \\ \downarrow & & \downarrow \\ \pi_1\bar{Y} & \to & \pi_1Y \end{array} \tag{7}$$

Suppose that f induces surjections on the fundamental groups of pure strata. Then $\bar{f}_*:\pi_1\bar{X}\to\pi_1\bar{Y}$ is surjective. Moreover, we may assume by induction that $f_{0*}:\pi_1X_0\to\pi_1Y_0$ is surjective. Therefore, we conclude that $f_*:\pi_1X\to\pi_1Y$ is surjective.

To prove the isomorphism part of the statement, we observe that the homotopy stratifications of X_0 and Y_0 induce homotopy stratifications of E_X and E_Y, such that the restrictions of f_E on pure strata are the fibration maps over the corresponding restrictions of f_0. The assumption that f_0 induces isomorphisms between the fundamental groups of the pure strata of X_0 and Y_0 implies that f_{E*} induces surjections (not necessarily isomorphisms!) between the fundamental groups of the pure strata of E_X and E_Y. By the surjectivity part of the proposition that we just proved, $f_{E*}:\pi_1E_X\to\pi_1E_Y$ is surjective.

Now in the map (7) between pushouts, f_{E*} is surjective, $\bar{f}_*$ is assumed to be isomorphic, and f_{0*} may be assumed to be isomorphic by induction. It then follows that $f_*:\pi_1X\to\pi_1Y$ is isomorphic.

Remark 1.6 Strictly speaking, if some spaces in (35) are not connected, then we need to use the term groupoid instead of groups in the proof. In the language of fundamental groups, there may be some extra generators and relations that produce HNN-extensions.

The one-to-one correspondence between the components makes these HNN-extensions equivalent for X and Y.

The following useful result says that isovariant $\pi-\pi$ structures may be restricted and induced.

Proposition 1.7 *Suppose that a homotopically G-stratified space X is an isovariant $\pi-\pi$ structure. Then for any subgroup H of G, X is an isovariant $\pi-\pi$ structure as an H-space, and X^H is an isovariant $\pi-\pi$ structure as a WH-space.*

Proof: The statement about X^H follows from the definition. In proving the statement about X as an H-space, one faces the complication of writing down the isovariant components of the H-action in terms of the components of the G-action:

$$X_{K,\text{for } H\text{-action}} = \cup_{K'\cap H=G} X_{K',\text{for } G\text{-action}}, \qquad K \subset H. \tag{8}$$

As in the proof of the proposition 1.4, we think of the expression (8) as a stratification of the isovariant components of the H-space X, with certain $X_{K',\text{for } G\text{-action}}$ as pure strata. Then by the proposition 1.5, the inclusion

$$X^{\partial\beta}_{K,\text{for } H\text{-action}} \subset (X^{\partial\beta} \cup X^{\beta})_{K,\text{for } H\text{-action}}$$

induces an isomorphism between fundamental groups.

Now we are ready to present our definition of the equivariant periodicity spaces.

Definition 1.8 A *G-periodicity space* with *periodicity representation V* is a homotopically G-stratified space P satisfying the following properties:

1. The representation disk DV is a closed stratum of P. Moreover, a closed union of DV with some other strata of P is a closed G-manifold Q after forgetting the stratification. Q has the property that for any subgroup $H \subset G$, the connected component of Q^H that contains DV^H is a WH-manifold with $\pi_0 WH$-signature 1 and $\pi_0 WH$-euler characteristic an odd number;

2. $\overline{P - DV}$ is a G-stratified isovariant $\pi - \pi$ structure.

Remark 1.9 The definition is very likely to be superseded by better ones as more technical machineries are available. Here we simply specify the basic properties our machinery requires.

By making use of the proposition 1.4, it is easy to prove the following result.

Lemma 1.10 *The products of periodicity spaces are periodicity spaces.*

It also immediately follows from our definition and the proposition 1.7 that periodicity spaces can be induced and restricted.

Lemma 1.11 *If P is a G-periodicity space with G-periodicity representation V, then*

1. *For any homomorphism $G' \to G$, the induced action makes P a G'-periodicity space with the induced G'-periodicity representation V;*

2. *For any subgroup $H \subset G$, P^H is a WH-periodicity space with WH-periodicity representation V^H.*

2 Periodicity Spaces for $SU(2)$ and $O(2)$

We will make the free use of the following facts in constructing the periodicity spaces for $SU(2)$ and $O(2)$.

S^1 is the circle group of complex numbers of norm 1. $SO(2) \cong S^1$ and the natural representation of $SO(2)$ on $\mathbf{R}^2 = \mathbf{C}$ is the same as the scalar multiplication by complex numbers in S^1.

Similarly, let S^3 be the Lie group of quaternionic numbers of norm 1. Then $SU(2) \cong S^3$ and the natural representation of $SU(2)$ on $\mathbf{C}^2 = \mathbf{H}$ is the same as the scalar multiplication *on the right* by quaternions in S^3.

Motivated by (2), we consider the $SU(2)$-stratified space

$$P = \mathbf{HP}^2 \cup D^5, \tag{9}$$

where D^5 is attached to $\mathbf{HP}^2$ via the identification $S^4 = \mathbf{HP}^1$. The action of $SU(2) = S^3$ is trivial on D^5, and by the formula

$$\lambda[z_1, z_2, z_3] = [z_1\lambda, z_2\lambda, z_3] \tag{10}$$

on $\mathbf{HP}^2$.

By $\mathbf{HP}^2 = D^8 \cup_{S^7 \nearrow} S^4$ (the map $S^7 \to S^4$ is the quaternionic Hopf bundle map). P is an $SU(2)$-stratified space with three strata

$$P = D^8 \cup_{S^7 \nearrow} S^4 {}_{\nwarrow S^4} \cup D^5, \tag{11}$$

where $SU(2)$ acts on $D^8 \subset \mathbf{C}^4$ (the first two coordinates of (10)) via twice of the natural representation of $SU(2)$ on $\mathbf{C}^2$, and acts trivially on D^5 and S^4. By taking $Q = \mathbf{HP}^2$, it is easy to see that P is $SU(2)$-periodic.

Lemma 2.1 *(11) is an $SU(2)$-periodicity space with twice of the natural $SU(2)$-representation as the periodicity representation.*

Our $O(2)$-periodicity space is much more complicated. Its stratification contains four isovariant $\pi - \pi$ pairs.

Let

$$V = \mathbf{H} \otimes_{\mathbf{R}} \mathbf{R}^2 = \mathbf{H}^2$$

be the quaternionization of the natural representation of $O(2)$ on $\mathbf{R}^2$. Motivated by the way the periodicity spaces are constructed for $S^1 = SO(1)$ and $SU(2)$, we start with the projective space

$$\mathbf{CP}(V \oplus \mathbf{C}) = DV \cup_{SV \nearrow \text{Hopf}} \mathbf{CP}(V) \tag{12}$$

with the induced $O(2)$-action. Next we list all the subgroups of $O(2)$ and study their fixed points in $\mathbf{CP}(V \oplus \mathbf{C})$.

The group $O(2)$ consists of two types of elements: The rotations

$$\rho_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \theta \in \mathbf{R} \mathrm{mod} 2\pi$$

by angle θ and the flippings

$$\tau_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \qquad \theta \in \mathbf{R} \mathrm{mod} \pi$$

with respect to the subspace $\rho_\theta(\mathbf{R} \oplus 0)$.

The only nontrivial infinite closed subgroup of $O(2)$ is the circle of rotations:

$$S^1 = SO(2) = \{\rho_\theta\}.$$

The finite subgroups of $O(2)$ are the cyclic subgroups of rotations:

$$C_n =< \rho_{\frac{2\pi}{n}} >, \quad n > 1,$$

the 2 element subgroups of flippings:

$$B_\theta =< \tau_\theta >= \{1, \tau_\theta\},$$

and the dihedral subgroups of rotations and flippings:

$$A_{\theta,n} =< \rho_{\frac{2\pi}{n}}, \tau_\theta >= B_\theta \propto C_n, \quad n > 1.$$

The normalizers and the Weyl groups of the subgroups are

$$\begin{array}{ll} N(S^1) = O(2), & W(S^1) = \{1, \bar{\tau}_0\} \cong \mathbf{Z}_2; \\ N(C_n) = O(2), & W(C_n) = O(2)/C_n \cong O(2); \\ N(B_\theta) = A_{\theta,2}, & W(B_\theta) = \{1, -1\} \cong \mathbf{Z}_2; \\ N(A_{\theta,n}) = A_{\theta,2n}, & W(A_{\theta,n}) = \{1, \bar{\rho}_{\frac{\pi}{n}}\} \cong \mathbf{Z}_2. \end{array}$$

The fixed points of the subgroups are described by the following proposition.

Proposition 2.2 *Suppose that W is a unitary complex H-representation. Then*

$$\mathbf{CP}(W)^H = \coprod_\kappa \mathbf{CP}(W^\kappa),$$

where

$$W^\kappa = \{w \in W : hw = \kappa(h)w \text{ for all } h \in H\}$$

is the eigenspace of a homomorphism (complex character) $\kappa : H \to S^1$.

We also observe that

$$\mathbf{CP}(W \oplus \mathbf{C})^H = \mathbf{CP}(W^H \oplus \mathbf{C}) \coprod [\coprod_{\kappa \text{ nontrivial}} \mathbf{CP}(W^\kappa)]. \tag{13}$$

$\mathbf{CP}(W^H \oplus \mathbf{C})$ is the component of $\mathbf{CP}(W \oplus \mathbf{C})^H$ containing DW^H. If the group action preserves orientation and $\dim_{\mathbf{C}} W^H$ is always even (e.g., W is twice of some other complex representation), then the equivariant signature of $\mathbf{CP}(W^H \oplus \mathbf{C})$ is 1, and the equivariant euler characteristic is $1 + \dim_{\mathbf{C}} W^H$, an odd number. What prevents $\mathbf{CP}(W \oplus \mathbf{C})$ to become a periodicity manifold is the potential existence of nontrivial "eigencharacter" κ, making the fixed points nonconnected.

ρ_θ has two complex eigenvalues $e^{i\theta}$, $e^{-i\theta}$ for the representation V. If $\theta \neq 0 \bmod \pi$, then the eigenspaces in V are respectively

$$(1,i)\mathbf{H} = \{(w, iw) : w \in \mathbf{H}\}, \quad (1,-i)\mathbf{H} = \{(w, -iw) : w \in \mathbf{H}\}.$$

Note that these are independent of the choice of the angle. Therefore they are actually the eigenspaces of the rotation subgroup S^1, corresponding to the two characters:

$$id : SO(2) \to S^1, \quad \rho_\theta \mapsto e^{i\theta}; \quad conj : SO(2) \to S^1, \quad \rho_\theta \mapsto e^{-i\theta}.$$

By (13),

$$\mathbf{CP}(V \oplus \mathbf{C})^{S^1} = \mathbf{CP}(V \oplus \mathbf{C})^{C_n} = 0 \coprod \mathbf{CP}((1,i)\mathbf{H}) \coprod \mathbf{CP}((1,-i)\mathbf{H}), \quad n > 2. \tag{14}$$

The action of the Weyl group $W(S^1) = \mathbf{Z}_2$ fixes 0 and exchanges the other two components. The action of the Weyl group $W(C_n) = O(2)$ factors through $O(2) \to \mathbf{Z}_2$.

τ_θ has two eigenvalues 1 and -1, with corresponding eigenspaces in V:

$$\mathbf{H}_\theta = \rho_\theta(\mathbf{H} \oplus 0), \quad \mathbf{H}_{\theta+\frac{\pi}{2}} = \rho_{\theta+\frac{\pi}{2}}(\mathbf{H} \oplus 0).$$

Thus by (13) again,

$$\mathbf{CP}(V \oplus \mathbf{C})^{B_\theta} = \mathbf{CP}(\mathbf{H}_\theta \oplus \mathbf{C}) \coprod \mathbf{CP}(\mathbf{H}_{\theta+\frac{\pi}{2}}). \tag{15}$$

The Weyl group $\mathbf{Z}_2$ acts by $-1 \oplus 1$ on the first component and trivially on the second component.

The fixed points of $A_{\theta,n}$, $n > 2$, are the intersection of the fixed points of C_n, $n > 2$, and those of B_θ. However, the intersections

$$\mathbf{CP}((1,i)\mathbf{H}) \cap \mathbf{CP}(\mathbf{H}_\theta \oplus \mathbf{C}), \qquad \mathbf{CP}((1,-i)\mathbf{H}) \cap \mathbf{CP}(\mathbf{H}_\theta \oplus \mathbf{C})$$

are easily seen to be empty. Therefore

$$\mathbf{CP}(V \oplus \mathbf{C})^{A_{\theta,n}} = 0, \quad n > 2. \tag{16}$$

The action of the Weyl group $\mathbf{Z}_2$ has to be trivial.

It remains to study the cyclic group C_2 and the 4 element groups $A_{\theta,2}$. We have

$$\mathbf{CP}(V \oplus \mathbf{C})^{C_2} = 0 \coprod \mathbf{CP}(V), \tag{17}$$

The action of the Weyl group $O(2)/C_2$ is trivial at 0 and is the one coming from the induced action of $O(2)$ on $\mathbf{CP}(V)$ (such action is trivial when restricted to C_2). Moreover, by taking the intersection of the fixed points of C_2 and B_θ, we obtain

$$\mathbf{CP}(V \oplus \mathbf{C})^{A_{\theta,2}} = 0 \coprod \mathbf{CP}(\mathbf{H}_\theta) \coprod \mathbf{CP}(\mathbf{H}_{\theta+\frac{\pi}{2}}). \tag{18}$$

The Weyl group $\mathbf{Z}_2$ acts trivially at 0 and exchanges the other two components.

Although the fixed point subspaces of $\mathbf{CP}(V \oplus \mathbf{C})$ are often not connected, the components not containing 0 are all contained in the submanifold $\mathbf{CP}(V)$. Therefore we may think of $\mathbf{CP}(V \oplus \mathbf{C})$ as a two strata space (12) with the equivariant submanifold $\mathbf{CP}(V)$ as the smaller stratum (later on we need to single out another submanifold $S^2 \times \{\pm 1\} \subset \mathbf{CP}(V)$, making $\mathbf{CP}(V \oplus \mathbf{C})$ a three strata space). If there is an $SU(2)$-manifold E with $\mathbf{CP}(V)$ as boundary, such that the pair $(E, \mathbf{CP}(V))$ satisfies the isovariant $\pi - \pi$ condition (see the definition 1.1), then $DV \cup_{SV \to \mathbf{CP}(V)} E$ is an $SU(2)$-periodicity space.

Our attempt of constructing E makes use of the quaternionic structure of the representation V. Consider the Hopf bundle map

$$S^2 \to \mathbf{CP}(V) \to \mathbf{HP}(V).$$

This is an equivariant $SU(2)$-bundle. Since the fibre is a sphere, the mapping cylinder

$$E = \mathbf{CP}(V) \times I \cup \mathbf{HP}(V)$$

is an $SU(2)$-manifold with boundary $\mathbf{CP}(V)$.

To check whether $(E, \mathbf{CP}(V))$ satisfies the isovariant $\pi - \pi$ condition, we need to know the structure of the fixed points in $\mathbf{HP}(V)$ and its relation to the structure of the fixed points of $\mathbf{CP}(V)$. First we need the quaternionic version of the proposition 2.2.

Proposition 2.3 *Suppose that W is a unitary quaternionic H-representation. Then*

$$\mathbf{HP}(W)^H = \coprod_{\text{conj. class of } \kappa} \mathbf{HP}(W^\kappa),$$

where

$$W^\kappa = \{w \in V : hw = \kappa(h)w \text{ for all } h \in H\}$$

is the set of eigenvectors of a homomorphism (quarternionic character) $\kappa : H \to S^3$.

Remark 2.4 If $hw = \kappa w$, then $h(\lambda w) = \lambda\kappa\lambda^{-1}(\lambda w)$. Therefore by the noncommutativity of the quaternionic multiplication, W^κ may not be a quaternionic subspace of W. However, we do have

$$\mathbf{H}W^\kappa = \cup_\lambda W^{\lambda\kappa\lambda^{-1}}.$$

The quaternionic projective space of W^κ then makes sense for the conjugacy class of κ.

The general relation between $\mathbf{CP}(W)^H$ and $\mathbf{HP}(W)^H$ is not very clear. However, we observe that if H is generated by a single element h, then the character κ is given by a complex or quaternionic eigenvalue a of h, and we will use the notation

$$W^{h=a} = W^\kappa = \{w \in W : hw = aw\}.$$

As further pointed out in the remark above, only the conjugacy class of the eigenvalue matters in the quaternionic case. The following proposition summarizes the facts in this respect.

Proposition 2.5 *The conjugacy action of $S^3 \subset \mathbf{H}$ on itself has the following properties:*
(1) *The action has two fixed points: the real numbers* 1 *and* -1*;*
(2) *Any nonreal quaternion is conjugate to exactly two complex numbers a and $\bar{a}$;*
(3) *The isotropy subgroup of any nonreal complex number is S^1, the complex numbers in S^3.*

We are now ready to study the relation between $\mathbf{CP}(W)^h$ and $\mathbf{HP}(W)^h$.

Proposition 2.6 *The projection $\mathbf{CP}(W)^h \to \mathbf{HP}(W)^h$ is surjective. A component of $\mathbf{HP}(W)^h$ is of the form $\mathbf{HP}(W^{h=a})$ for some complex number a. The components of $\mathbf{CP}(W)^h$ over $\mathbf{HP}(W^{h=a})$ are the following:*

1. *If a is real, then $W^{h=a}$ is a quaternionic subspace of W, and there is only one corresponding component $\mathbf{CP}(W^{h=a})$. Moreover, $\mathbf{CP}(W^{h=a}) \to \mathbf{HP}(W^{h=a})$ is a Hopf bundle with S^2 fibre;*

2. *If a is not real, then there are exactly two corresponding components $\mathbf{CP}(W^{h=a})$ and $\mathbf{CP}(W^{h=\bar{a}})$. Moreover, the projections $\mathbf{CP}(W^{h=a}) \to \mathbf{HP}(W^{h=a}) \leftarrow \mathbf{CP}(W^{h=\bar{a}})$ are homeomorphic.*

Proof: By the proposition 2.3, a component of $\mathbf{HP}(W)^h$ is of the form $\mathbf{HP}(W^{h=b})$ for some quaternion b. By the proposition 2.5, b is conjugate to a complex number a. Thus by the remark above, $\mathbf{HP}(W^{h=b}) = \mathbf{HP}(W^{h=a})$. This proves the statement about the components of $\mathbf{HP}(W)^h$.

The components of $\mathbf{CP}(W)^h$ over $\mathbf{HP}(W^{h=a})$ have to be of the form $\mathbf{CP}(W^{h=c})$ with c being a complex number conjugating to a via quaternions. By the first and the second properties of the proposition 2.5, we have the two situations listed in this proposition.

If a is real (i.e., $a = 1$ or -1), then the fact that $W^{h=a}$ is a quaternionic space implies that the projection $\mathbf{CP}(W^{h=a}) \to \mathbf{HP}(W^{h=a})$ is a Hopf bundle.

If a is not real, then we need to show that the projection $\mathbf{CP}(W^{h=a}) \to \mathbf{HP}(W^{h=a})$ is homeomorphic. It is obviously surjective. For the injectivity, we consider vectors $u, v \in W^{h=a}$ of length 1, such that $[u]_{\mathbf{H}} = [v]_{\mathbf{H}} \in \mathbf{HP}(W^{h=a})$. Thus $v = \lambda u$ for some quaternion $\lambda \in S^3$. Therefore $\lambda a u = \lambda h u = h \lambda u = hv = av = a\lambda u$. Since $u \neq 0$, we see that $\lambda^{-1} a \lambda = a$. By the third property of the proposition 2.5, λ is a complex number. Consequently, $[u]_{\mathbf{C}} = [\lambda u]_{\mathbf{C}} = [v]_{\mathbf{C}} \in \mathbf{CP}(W^{h=a})$. This proves the injectivity.

Finally, the projection $\mathbf{CP}(W)^h \to \mathbf{HP}(W)^h$ is surjective because in both cases, the projections on components are surjective.

Now we are ready to study the fixed points in E.

First we observe that $C_2 = \{1, -1\}$ fixes E. Therefore the action of $O(2)$ factors through $O(2)/C_2 \cong O(2)$.

If $\theta \neq 0 \mathrm{mod} \pi$, then the two complex eigenvalues $e^{i\theta}$, $e^{-i\theta}$ of ρ_θ are not real. Therefore the fixed points of $C_n = < \rho_{\frac{2\pi}{n}} >$, $n > 2$, and S^1 are described by the second case of the proposition 2.6. The following proposition describes the S^1-fixed points of E. The same description applies to the fixed points of C_n, $n > 2$, with the action of the Weyl group factoring through $O(2) \to \mathbf{Z}_2$.

Proposition 2.7 *$E^{S^1} \cong S^2 \times [-1,1]$, with the action of the Weyl group $\mathbf{Z}_2$ given by the antipodal on S^2 and the flipping $t \mapsto -t$ on the interval.*

Proof: E^{S^1} is the mapping cylinder of

$$\begin{array}{rl} & \mathbf{CP}(V)^{S^1} = \mathbf{CP}((1,i)\mathbf{H}) \coprod \mathbf{CP}((1,-i)\mathbf{H}) \\ \to & \mathbf{HP}(V)^{S^1} = \mathbf{HP}((1,i)\mathbf{H}) = \mathbf{HP}((1,-i)\mathbf{H}), \end{array}$$

From the natural identifications

$$\mathbf{HP}((1,i)\mathbf{H}) \cong \mathbf{CP}((1,i)\mathbf{H}) \cong \mathbf{CP}(\mathbf{H}) = S^2, \quad [w, iw] \to [w], \tag{19}$$

$$\mathbf{HP}((1,-i)\mathbf{H}) \cong \mathbf{CP}((1,-i)\mathbf{H}) \cong \mathbf{CP}(\mathbf{H}) = S^2, \quad [w, -iw] \to [w], \tag{20}$$

we see that

$$E^{S^1} = S^2 \times [-1,1], \tag{21}$$

where we put $\mathbf{CP}(V)^{S^1}$ at the two ends $S^2 \times \{\pm 1\}$ and $\mathbf{HP}(V)^{S^1}$ at the middle $S^2 \times 0$.

As for the action of the Weyl group $W(S^1) = \mathbf{Z}_2$, we observe that the action is the flipping $t \to -t$ on the interval $[-1,1]$ because the Weyl group exchanges the two components $\mathbf{CP}((1,i)\mathbf{H})$ and $\mathbf{CP}((1,-i)\mathbf{H})$. In the S^2 direction, we consider the action at the level $t = 0$:

$$\tau_0[w, iw] = [w, -iw] = [jw, i(jw)] \in \mathbf{HP}((1,i)\mathbf{H}).$$

If we use the identification (19), then

$$\tau_0[w] = [jw] \in \mathbf{CP}(\mathbf{H}) = S^2.$$

This is the antipodal map.

The eigenvalues of τ_θ over V are 1 and -1. Therefore the fixed points of $B_\theta =< \tau_\theta >$ are described by the first case of the proposition 2.6. In particular, $\mathbf{H}_\theta$ is always a quaternionic subspace of V, and E^{B_θ} is the mapping cylinder of

$$\begin{array}{rl} & \mathbf{CP}(V)^{B_\theta} = \mathbf{CP}(\mathbf{H}_\theta) \coprod \mathbf{CP}(\mathbf{H}_{\theta+\frac{\pi}{2}}) \\ \to & \mathbf{HP}(V)^{B_\theta} = \mathbf{HP}(\mathbf{H}_\theta) \coprod \mathbf{HP}(\mathbf{H}_{\theta+\frac{\pi}{2}}). \end{array}$$

In fact, the projection is simply $S^2 \coprod S^2 \to 2$ points. Therefore

$$E^{B_\theta} = D^3 \coprod D^3. \tag{22}$$

The action of the Weyl group $\mathbf{Z}_2$ is trivial.

The fixed points of $A_{\theta,n}$ is the intersection of the fixed points of C_n and B_θ. Because $\mathbf{HP}((1,i)H) \cap \mathbf{HP}(\mathbf{H}_\theta)$ and $\mathbf{HP}((1,-i)H) \cap \mathbf{HP}(\mathbf{H}_\theta)$ are easily seen to be empty, we have

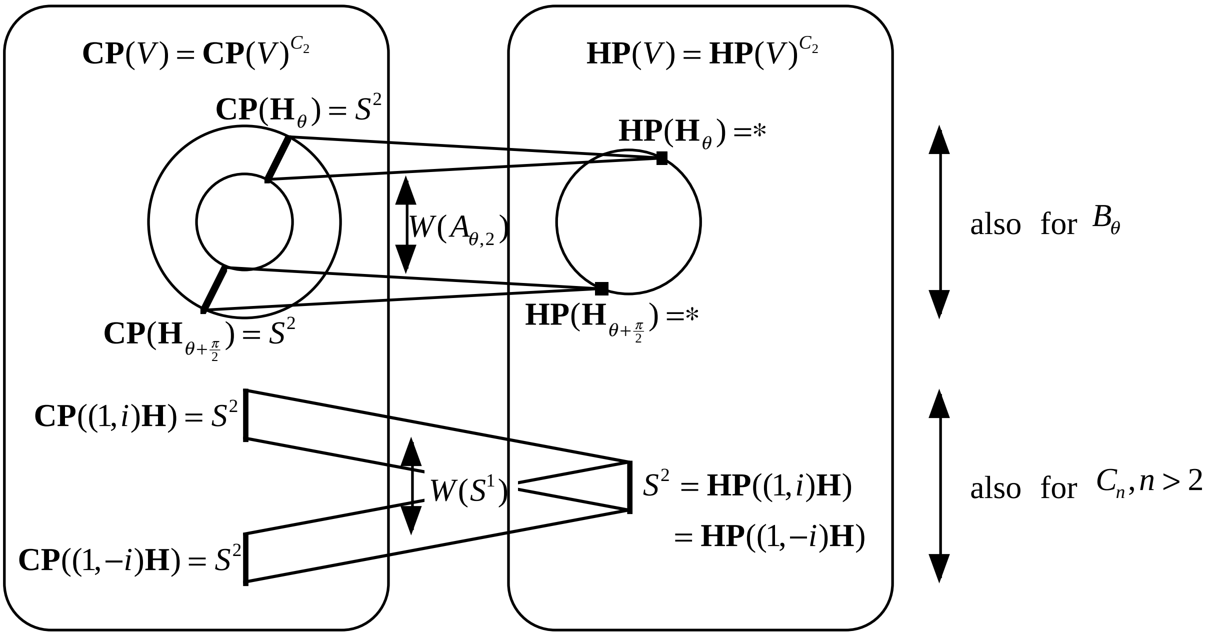
$$E^{A_{\theta,n}} = \emptyset, \quad n > 2. \tag{23}$$

Moreover, since C_2 acts trivially on $\mathbf{CP}(V)$,

$$E^{A_{\theta,2}} = E^{B_\theta} = D^3 \coprod D^3. \tag{24}$$

The Weyl group $\mathbf{Z}_2$ of $A_{\theta,2}$ exchanges the two disks (in contrast to the behavior of the Weyl group of B_θ).

The following is the picture of all fixed points in E.



In order that the $O(2)$-stratified space $P_1 = \mathbf{CP}(V\oplus\mathbf{C})\cup_{\mathbf{CP}(V)} E$ to be periodic, $(E,\partial E)=(E,\mathbf{CP}(V))$ has to satisfy the isovariant $\pi-\pi$ condition. We see from the above picture that $(E,\partial E)$ has codimension ≥ 3 gap. Therefore the inclusion from the isovariant components to the equivariant components is one-to-one, and the corresponding components have isomorphic fundamental groups. Consequently, the isovariant $\pi-\pi$ condition is the same as the analogous equivariant $\pi-\pi$ condition. In what follows, we check such equivariant condition.

First of all, the whole space E is fixed by the subgroup $C_2=\{1,-1\}$. Both E and ∂E have trivial fundamental groups. Hence the equivariant $\pi-\pi$ condition is satisfied by C_2.

Second, the fixed points $E^{B_\theta}=E^{A_{\theta,2}}$ consists of two components, each of which is homeomorphic to D^3, with boundary homeomorphic to S^2. Both have trivial fundamental groups. Hence the equivariant $\pi-\pi$ condition is satisfied by B_θ and $A_{\theta,2}$.

Third, the equivariant $\pi-\pi$ condition is trivially satisfied by $A_{\theta,n}$, $n>2$, since no points in E is fixed by the subgroup.

The problem arises with the fixed points of S^1 and C_n, $n>2$. The proposition 2.7 shows that $E^{S^1}=E^{C_n}=S^2\times[-1,1]$ is connected, while $\partial E^{S^1}=\partial E^{C_n}=S^2\times\{\pm1\}$ has two components.

The failure of the $\pi-\pi$ condition for $(E,\partial E)^{S^1}=(E,\partial E)^{C_n}$ may be fixed by a $\mathbf{Z}_2$-equivariant nullcobordism (F,F^∂) of the pair $(E,\partial E)^{S^1}$ such that both (F,E^{S^1}) and

$(F^\partial, \partial E^{S^1})$ satisfy the isovariant $\pi - \pi$ condition. With such (F, F^∂),

$$P_2 = P_1 \cup F = \mathbf{CP}(V \oplus \mathbf{C}) \cup_{\mathbf{CP}(V)} E \cup_{E^{S^1}} F.$$

is a periodicity space (Strictly speaking, we should take the union of P_1 with $O(2) \times_{N(S^1)} F$. However, $O(2) = N(S^1)$.).

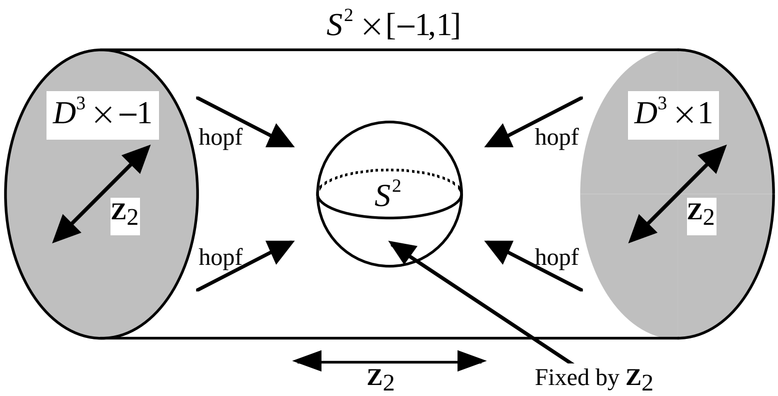
We choose $F^\partial = D^3 \times \{\pm 1\}$ as the $\mathbf{Z}_2$-equivariant nullcobordism of $\partial E^{S^1} = S^2 \times \{\pm 1\}$. Then $E^{S^1} \cup_{\partial E^{S^1}} F^\partial = S^2 \times [-1,1] \cup D^3 \times \{\pm 1\}$ is a sphere S^3, and the $\mathbf{Z}_2$-action is antipodal. Therefore we may consider the Hopf bundle

$$S^1 \to S^3 \to S^2.$$

This is a principal S^1-bundle, with the antipodal action on S^3 induced by the fibrewise S^1-action. The associated disc bundle

$$F = S^3 \times I \cup S^2$$

is then a $\mathbf{Z}_2$-equivariant nullcobordism of the antipodal S^3.



Clearly, $(F^\partial, \partial E^{S^1}) = (D^3, S^2) \times \{\pm 1\}$ is a $\mathbf{Z}_2$-isovariant $\pi - \pi$ structure.

As for the pair (F, E^{S^1}), we have

$$F^{\mathbf{Z}_2} = S^2 \neq \emptyset = (E^{S^1})^{\mathbf{Z}_2}.$$

On the $\mathbf{Z}_2$-free part, however,

$$F - F^{\mathbf{Z}_2} = S^3 \times [0,1) \supset S^2 \times [-1,1] = E^{S^1} - (E^{S^1})^{\mathbf{Z}_2}$$

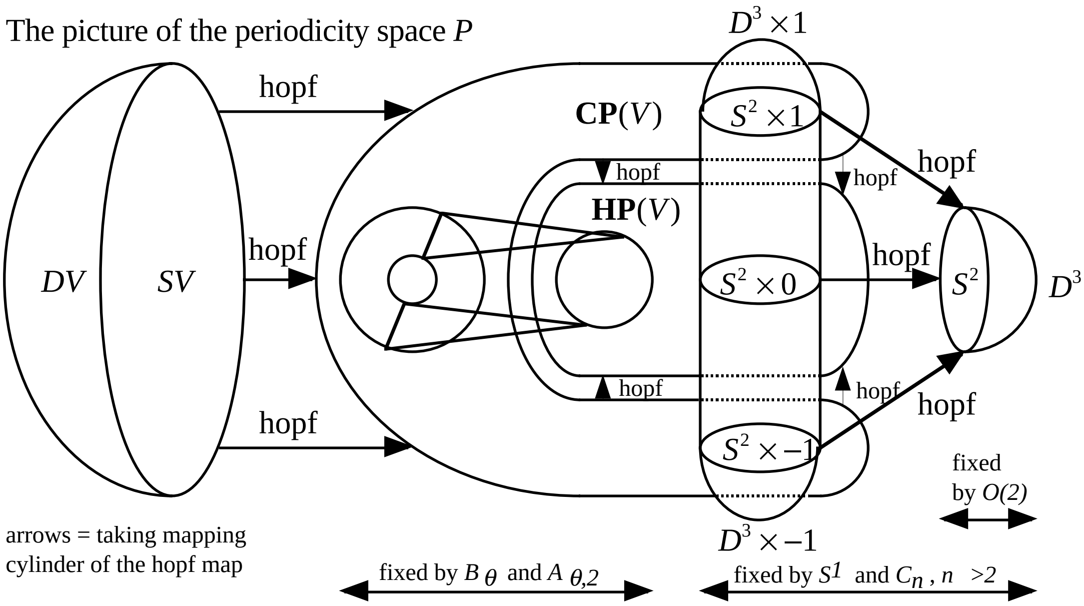
are all connected and simply connected. Thus the free part of (F, E^{S^1}) satisfies the isovariant $\pi - \pi$ condition, while the nonfree part fails to satisfy.

The failure on the nonfree part may again be fixed by another nullcobordism D^3 of $F^{\mathbf{Z}_2} = S^2$. The pair (D^3, S^2) satisfies the $\pi - \pi$ condition.

In conclusion, we build our periodicity space by successively equivariantly nullcobording fixed point components, eliminating all the nullcobordisms that do not satisfy the $\pi - \pi$ condition. The space we end up with is (recall $V = \mathbf{H}^2$)

$$\begin{aligned} P = \mathbf{CP}(V \oplus \mathbf{C}) \ & \cup_{\mathbf{CP}(V)} (\mathbf{CP}(V) \times I \cup \mathbf{HP}(V)) \\ & \cup_{S^2 \times [-1,1]} (S^3 \times I \cup S^2) \\ & \cup_{S^2} D^3, \end{aligned} \tag{25}$$

The action of $O(2)$ on P is the induced one on the projective spaces. It is the antipodal action on S^3 via the homomorphism $O(2) \to \mathbf{Z}_2$. The action is trivial on D^3.



The stratification of P is illustrated by the following diagram.

$$\begin{array}{ccccccc}
\mathbf{CP}(V \oplus \mathbf{C})_{(5)} & & & & & & \\
\cup & & & & & & \\
\mathbf{CP}(V)_{(\partial 4)} & \subset & E_{(4)} & & S^2_{(\partial 2)} & \subset & D^3_{(2)} \\
& & \cup & & \cap & & \\
& & E^{S^1}_{(\partial 3)} & \subset & F_{(3)} & & \\
& & \cup & & \cap & & \\
& & \partial E^{S^1}_{(\partial 1)} & \subset & F^{\partial}_{(1)} & &
\end{array}$$

where

$$\begin{aligned}
&\mathbf{CP}(V \oplus \mathbf{C}) = DV \cup_{SV \nearrow \mathrm{Hopf}} \mathbf{CP}(V), \\
&E = \mathbf{CP}(V) \times I \cup_{\mathrm{Hopf}} \mathbf{HP}(V), \\
&E^{S^1} = S^2 \times [-1, 1], \\
&\partial E^{S^1} = S^2 \times \{\pm 1\}, \\
&F = S^3 \times I \cup_{\mathrm{Hopf}} S^2, \\
&F^{\partial} = D^3 \times \{\pm 1\}.
\end{aligned}$$

The terms in the diagram are closed strata. The numerical footnotes denote the indices. The index set is obtained from the double of $1 < 2 < 3 < 4 < 5$ by deleting $\partial 5$.

Lemma 2.8 *(25) is an $O(2)$-periodicity space with the four fold natural $O(2)$-representation as the periodicity representation.*

3 Proof of the Periodicity

The theorem 0.1 is special cases of the following theorem.

Theorem 3.1 *Suppose that M is a homotopically stratified G-manifold with codimension ≥ 3 gap. Suppose that P is a G-periodicity space with periodicity representation V, such that M and $M \times V$ have the same isotropy everywhere. Then there is a natural equivalence*

$$S_G(M, \text{rel } \partial M) \cong S_G(M \times DV, \text{rel } \partial(M \times DV)).$$

Remark 3.2 The periodicity map in the theorem may depend on more than the representation V. If the $\pi - \pi$ structures in the periodicity spaces are cobordant through $\pi - \pi$ structures (of one dimension higher), then the corresponding periodicity maps are homotopy equivalent.

Remark 3.3 The periodicity equivalence is natural with respect to transverse isovariant G-maps, and the restriction to fixed points of subgroups. Such naturalities follow from the formal nature of the proof of the Theorem.

Remark 3.4 For the special periodicity space for the abelian group actions, it was further proved in [WY] that the periodicity is natural with respect to the induction $S_G(M) \to S_{G'}(M)$ (provided that $[G : im(G')]$ is finite) and the restriction $S_G(M) \to S_{WH}(M^H)$. Such naturality remains true in general.

The proof of the theorem 3.1 is similar to the one in [WY].

We will use spacified version of the surgery theory. In the classical case, this was done in [Q1]. The formalism may be adopted to the isovariant and more generally, stratified case (see [We] for more details). Therefore $L(X)$ will be a space whose homotopy groups are the surgery obstructions groups of X at various dimensions (In particular, the proposition 1.2 may be interpreted as the contractibility of the space $L_G(X)$ for a G-isovariant $\pi - \pi$ structure X). Similarly, $S(X)$ will be a space whose homotopy groups are the structure sets of $X \times \Delta^i$ relative to the boundary for various i. We will also use $K^{\leq 1}$ to denote an involutive spectrum with

$$\pi_i K^{\leq 1} = \begin{cases} 0 & i > 1 \\ \text{whitehead torsion} & i = 1 \\ \text{finiteness obstruction} & i = 0 \\ \text{negative } K\text{-obstruction} & i < 0 \end{cases}$$

Finally we note that long exact sequences of obstruction groups are usually the long exact sequences of homotopy groups of fibrations of obstruction spaces.

According to Weinberger [We], the computation of the isovariant structure $S_G(M, \text{rel } \partial)$ involves several fibrations. First we compute the stable isovariant structure through the surgery fibration:

$$S_G^{-\infty}(M, \text{rel } \partial) \to H(M/G; L_G^{-\infty}(locM)) \to L_G^{-\infty}(M). \tag{26}$$

$L_G^{-\infty}$ is the stablization of Browder-Quinn's isovariant surgery obstruction [BQ]. The difference between L_G and $L_G^{-\infty}$ is given by the Rothenberg fibration

$$L_G \to L_G^{-\infty} \to \hat{H}(\mathbf{Z}_2; K_G^{\leq 1}). \tag{27}$$

$K_G^{\leq 1}$ is Browder-Quinn's isovariant K-obstruction [BQ]. $\hat{H}(\mathbf{Z}_2; ?)$ is the (spacified) Tate cohomology, which may be applied to involutive spectra and converts an involutive fibration into a fibration.

The unstable structure $S_G(M, \mathrm{rel}\ \partial)$ can then be computed from another Rothenberg fibration

$$S_G \to S_G^{-\infty} \to \hat{H}(\mathbf{Z}_2; Wh_G^{top, \leq 0}). \tag{28}$$

$Wh_G^{top, \leq 0}$ is the topological isovariant K-obstruction [Q2][S]. It is related to $K_G^{\leq 1}$ in a way similar to the stable surgery fibration (26)

$$Wh_G^{top}(M) \to H(M/G; K_G^{\leq 1}(locM)) \to K_G^{\leq 1}(M). \tag{29}$$

There is however one catch: $Wh_G^{top, \leq 0}$ is the truncation of Wh_G^{top} at dimension 0. This troublesome catch is responsible for our requirement on the equivariant euler characteristic in the definition of periodicity spaces.

The theorem 3.1 is proved by showing successively that

$$M \overset{\times P}{\Longrightarrow} M \times P \overset{\text{incl}}{\Longleftarrow} M \times (DV, \mathrm{rel}\ SV), \tag{30}$$

induces periodicity equivalences on L_G, $L_G^{-\infty}$, $S_G^{-\infty}$, and S_G.

Proof of the Periodicity on L_G:

Consider the maps

$$L_G(M) \xrightarrow{\times P} L_G(M \times P) \xleftarrow{\text{incl}} L_G(M \times DV, \mathrm{rel}\ SV)$$

induced by (30). The inclusion fits into a fibration

$$L_G(M \times DV, \mathrm{rel}\ SV) \xrightarrow{\text{incl}} L_G(M \times P) \to L_G(M \times \overline{P - DV}), \tag{31}$$

Since $\overline{P - DV}$ is an isovariant $\pi - \pi$ structure, we conclude from the propositions 1.2 and 1.4 that $L_G(M \times \overline{P - DV})$ is contractible. Therefore the inclusion is an equivalence.

The proof that $\times P$ also induces an equivalence is more complicated. We first consider the case that G acts on M freely. Construct the diagram

$$\begin{array}{ccc} L_G(M \times Q) & \xleftarrow{\text{incl}} & L_G(M \times DV, \mathrm{rel}\ SV) \\ \times_Q \uparrow & \nwarrow \phi & \downarrow \text{incl} \\ L_G(M) & \xrightarrow{\times P} & L_G(M \times P) \end{array} \tag{32}$$

where the map ϕ first restricts to $Q \subset P$ and then forgets the stratification structure inside Q. The two triangles are commutative by the geometric meaning of the maps. Because M is a free G-manifold, $\times Q$ may be identified with the map

$$L(M/G) \to L((M \times Q)/G)$$

of classical surgery obstructions obtained as the transfer of the bundle

$$Q \to (M \times Q)/G \to M/G. \tag{33}$$

Lück and Ranicki [LR] proved that such map depends only on the $\pi_0 G$-equivariant signature of Q. Since the signature is 1, $\times Q$ is an equivalence. Moreover, since G acts freely on M, the horizontal inclusion is the inclusion $L((M \times DV)/G) \subset L((M \times Q)/G)$ of classical surgery obstructions. By the simple connectivity of Q, we have $\pi_1((M \times DV)/G) = \pi_1(M/G) = \pi_1((M \times Q)/G)$. Consequently, the horizontal inclusion is an equivalence. Finally, we have already proved that the vertical inclusion is also an equivalence. Therefore we may conclude that $\times P$ at the bottom of the diagram is an equivalence.

For the general case, we induct on the isotropy subgroups of M. Let H be a maximal isotropy subgroup of M. Then we consider the diagram

$$\begin{array}{ccccc} L_G(M-GM^H) & \longrightarrow & L_G(M\times P-G(M^H\times P^H)) & \stackrel{\text{incl}}{\longleftarrow} & L_G(M\times DV-G(M^H\times DV^H),\text{rel } SV) \\ \downarrow & & \downarrow & & \downarrow \\ L_G(M) & \stackrel{\times P}{\longrightarrow} & L_G(M\times P) & \stackrel{\text{incl}}{\longleftarrow} & L_G(M\times DV,\text{rel } SV) \\ \downarrow & & \downarrow & & \downarrow \\ L_{WH}(M^H) & \stackrel{\times P^H}{\longrightarrow} & L_{WH}(M^H\times P^H) & \stackrel{\text{incl}}{\longleftarrow} & L_{WH}(M^H\times DV^H,\text{rel } SV) \end{array} \tag{34}$$

where the columns are fibrations. Observe that WH acts freely on M^H, and P^H is a WH-periodicity space by the lemma 1.11. We have just proved that the bottom maps are equivalences. Therefore in order to prove the middle maps are equivalences, it suffices to prove that the top maps are equivalences. We apply (30) to $M - GM^H$ and compare it with the top of (34):

$$\begin{array}{ccccc} L_G(M-GM^H) & \stackrel{\times P}{\longrightarrow} & L_G((M-GM^H)\times P) & \stackrel{\text{incl}}{\longleftarrow} & L_G((M-GM^H)\times DV,\text{rel } SV) \\ \| & & \downarrow \text{incl} & & \downarrow \text{incl} \\ L_G(M-GM^H) & \longrightarrow & L_G(M\times P-G(M^H\times P^H)) & \stackrel{\text{incl}}{\longleftarrow} & L_G(M\times DV-G(M^H\times DV^H),\text{rel } SV) \end{array} \tag{35}$$

By induction, we may assume that the upper row consists of equivalences. Therefore it remains to show that the vertical inclusions are equivalences.

The condition that M and $M \times DV$ have the same isotropy everywhere imply that the vertical inclusion on the right does not introduce any new strata or isovariant components and induce isomorphisms on the fundamental groups of corresponding isovariant components in pure strata. Therefore the right inclusion is an equivalence.

We have proved that the upper inclusion is an equivalence because $\overline{P - DV}$ is an isovariant $\pi - \pi$ structure. Similarly, the lower inclusion is an equivalence because both $\overline{P - DV}$ and $\overline{P^H - DV^H}$ are isovariant $\pi - \pi$ structures. As a consequence, the vertical inclusion at the middle is an equivalence.

The same isotropy everywhere condition and the codimension ≥ 3 condition imply that both vertical inclusions do not introduce any new strata or isovariant components and induce isomorphisms on the fundamental groups of isovariant components in pure strata. Such maps induce equivalences on the surgery obstructions.

Proof of the Periodicity on $L_G^{-\infty}$:

By Rothenberg fibration (27), the equivalences on the stable surgery obstruction $L_G^{-\infty}$ will follow from the compatible equivalences on L_G and $\hat{H}(\mathbf{Z}_2; K_G^{\leq 1})$. Thus we need to repeat the argument for the equivalence on L_G again, this time on $\hat{H}(\mathbf{Z}_2; K_G^{\leq 1})$.

$K^{\leq 1}$ is a functor over stratified (and in particular, equivariant) spaces. It has the following decomposition property: If Y is a closed union of strata of X, then there is a involutive fibration

$$K^{\leq 1}(X - Y) \to K^{\leq 1}(X) \to K^{\leq 1}(Y) \tag{36}$$

with a natural (and often noninvolutive) splitting $K^{\leq 1}(Y) \to K^{\leq 1}(X)$. Since the Tate cohomology functor $\hat{H}(\mathbf{Z}_2; ?)$ converts involutive fibrations to fibrations, the decomposition property implies that the functor $\hat{H}(\mathbf{Z}_2; K^{\leq 1})$ satisfies the second condition (decomposition along singularities) in the remark after the proposition 1.2. If we can show that $\hat{H}(\mathbf{Z}_2; K^{\leq 1})$ also satisfies the first condition in the remark (classical $\pi - \pi$ theorem, which $K^{\leq 1}$ alone does not satisfy), then the proposition 1.2 is also valid for $\hat{H}(\mathbf{Z}_2; K_G^{\leq 1})$.

For a manifold pair $(X, \partial X)$, we have

$$K^{\leq 1}(X, \partial X) = K^{\leq 1}(X, \mathrm{rel}\ \partial X) \times K^{\leq 1}(X), \tag{37}$$

with the involution

$$(\alpha, \beta)^* = (\alpha^* - (i_*\beta)^*, \beta^*), \tag{38}$$

where $i_* : K^{\leq 1}(\partial X) \to K^{\leq 1}(X, \mathrm{rel}\ \partial X)$ is the natural inclusion map and satisfies

$$(i_*\beta)^* = -i_*(\beta^*). \tag{39}$$

Suppose X, ∂X are connected, and $\pi_1 X = \pi_1 \partial X$. Since $K^{\leq 1}$ depends only on fundamental groups, we see that i_* is an equivalence. It was easy to prove (see [WY], for example) that the involution (38) in which i_* is an isomorphism satisfying (39) has to have the vanishing Tate cohomology. Therefore the classical $\pi - \pi$ theorem is valid for $\hat{H}(\mathbf{Z}_2; K^{\leq 1})$.

With the proposition 1.2 at hand, we are able to show that the inclusion in (30) induces an equivalence on $\hat{H}(\mathbf{Z}_2; K_G^{\leq 1})$, similar to the equivalence on L_G.

In further repeating the argument for the periodicity of L_G, we encounter the following difference between L_G and $K_G^{\leq 1}$: By Anderson [A], the transfer of the K-obstructions is given by multiplying the equivariant euler characteristic, instead of Lück and Ranicki's equivariant signature. In the definition of periodicity spaces, we have assumed that the equivariant euler characteristic is an odd number (perhaps different for different isotropy groups). Therefore the product with the number induces equivalences after localizing at 2. Since the Tate cohomology is 2-torsion, localization at 2 does not change the Tate cohomology. Therefore by working with $\hat{H}(\mathbf{Z}_2; K_G^{\leq 1} \otimes \mathbf{Z}_{(2)}) = \hat{H}(\mathbf{Z}_2; K_G^{\leq 1})$, we may still carry out the argument.

Despite the difference between L_G and $K_G^{\leq 1}$, we have all the properties we need to repeat the proof of the periodicity on L_G for the functor $\hat{H}(\mathbf{Z}_2; K_G^{\leq 1})$. Consequently, (30) induces equivalences on the stable isovariant surgery obstructions $L_G^{-\infty}$.

Proof of the Periodicity on $S_G^{-\infty}$:

To prove the periodicity for the stable structure $S_G^{-\infty}$, we view the periodicity on $L_G^{-\infty}$ as a natural equivalence between functors:

$$L_G^{-\infty}(?) \xrightarrow{\times P} L_G^{-\infty}(? \times P) \xleftarrow{\text{incl}} L_G^{-\infty}(? \times DV, \text{rel } SV).$$

We may take the assemblies of their homologies over M/G to obtain a commutative diagram

$$\begin{array}{ccc} H(M/G; L_G^{-\infty}(locM)) & \to & L_G^{-\infty}(M) \\ \simeq\downarrow \times P & & \simeq\downarrow \times P \\ H(M/G; L_G^{-\infty}((locM) \times P)) & \to & L_G^{-\infty}(M \times P) \\ \simeq\uparrow \text{ incl} & & \simeq\uparrow \text{ incl} \\ H(M/G; L_G^{-\infty}((locM) \times DV, \text{rel } SV)) & \to & L_G^{-\infty}(M \times DV, \text{rel } SV) \end{array} \tag{40}$$

The homotopy fibre of the top map is $S_G^{-\infty}(M, \text{rel } \partial M)$. To work out the homotopy fibre of the bottom map, we rewrite the homology:

$$\begin{array}{rl} & H(M \times DV/G; L_G(loc(M \times DV))) \\ \overset{\text{Fubini}}{\simeq} & H(M/G; H(DV/G_x; L_G(loc(M \times DV)))) \\ \overset{\alpha}{\simeq} & H(M/G; L_G((locM) \times DV, \text{rel } SV)). \end{array} \tag{41}$$

The Fubini equivalence is a basic property of the stratified homology. α is obtained by applying the homology to the natural transformation (of functors of $locM$)

$$\alpha_0 : H(DV/G_x; L_G^{-\infty}(loc(M \times DV))) \to L_G^{-\infty}((locM) \times DV).$$

α_0 is an assembly over a cone $DV/G_x = \text{cone}(SV/G_x)$, which is always an equivalence (lemma 3.21 of [Y]). Consequently α is an equivalence. By the naturality of the Fubini equivalence with respect to the assembly, the bottom of (40) may be identified with the assembly of $L_G^{-\infty}(?)$ over $(M \times DV)/G$. The homotopy fibre of this assembly is $S_G^{-\infty}(M \times DV, \text{rel } \partial(M \times DV))$. Hence we obtain the equivalence between $S_G^{-\infty}(M, \text{rel } \partial M)$ and $S_G^{-\infty}(M \times DV, \text{rel } \partial(M \times DV))$. This proves the periodicity of the stable structure.

Proof of the Periodicity on S_G:

The last step is destablization. By the Rothenberg fibration (28), the periodicity on S_G will follow from compatible periodicities on $S_G^{-\infty}$ and on $\hat{H}(\mathbf{Z}_2; Wh_G^{top,\leq 0})$. This is the most intricate step.

The first intricacy is the meaning of the periodicity on $\hat{H}(\mathbf{Z}_2; Wh_G^{top,\leq 0})$. We are not studying the maps

$$Wh_G^{top,\leq 0}(M) \overset{\times P}{\to} Wh_G^{top,\leq 0}(M \times P) \overset{\text{incl}}{\leftarrow} Wh_G^{top,\leq 0}(M \times DV, \text{rel } SV)$$

induced from (30). In fact, these maps generally do not induce equivalences on the Tate cohomologies.

To clarify our problem, we start by considering a diagram similar to (40)

$$\begin{array}{ccc} H(M/G; K_G^{\leq 1}(locM)) & \to & K_G^{\leq 1}(M) \\ \downarrow {\scriptstyle \times P} & & \downarrow {\scriptstyle \times P} \\ H(M/G; K_G^{\leq 1}((locM) \times P)) & \to & K_G^{\leq 1}(M \times P) \\ \uparrow {\scriptstyle \mathrm{incl}} & & \uparrow {\scriptstyle \mathrm{incl}} \\ H(M/G; K_G^{\leq 1}((locM) \times DV, \mathrm{rel}\ SV)) & \to & K_G^{\leq 1}(M \times DV, \mathrm{rel}\ SV) \end{array} \tag{42}$$

We also introduce the notation $Wh_G^{top}(M; A, \mathrm{rel}\ B)$ as the homotopy fibre of the assembly

$$H(M/G; K_G^{\leq 1}((locM) \times (A, \mathrm{rel}\ B))) \to K_G^{\leq 1}(M \times (A, \mathrm{rel}\ B)).$$

Then we have the induced maps between the homotopy fibres of the horizontal maps in (42):

$$Wh_G^{top}(M) \overset{\times P}{\to} Wh_G^{top}(M; P) \overset{\mathrm{incl}}{\leftarrow} Wh_G^{top}(M; DV, \mathrm{rel}\ SV). \tag{43}$$

As in the proof of the periodicity on $S_G^{-\infty}$, we may use Fubini equivalence to show that

$$Wh_G^{top}(M; DV, \mathrm{rel}\ SV) \simeq Wh_G^{top}(M \times DV, \mathrm{rel}\ SV). \tag{44}$$

Because the proof makes use of the fact that $DV = \mathrm{cone}(SV)$, we cannot conclude $Wh_G^{top}(M; P) = Wh_G^{top}(M \times P)$ (which is generally false). The situation is similar to what happens for the periodicity on $S_G^{-\infty}$. We may introduce a notation $S_G^{-\infty}(M; P)$ for the homotopy fibre of the assembly at the middle row of (40). Then we obtain equivalences

$$S_G^{-\infty}(M) \overset{\times P}{\simeq} S_G^{-\infty}(M; P) \overset{\mathrm{incl}}{\simeq} S_G^{-\infty}(M \times DV, \mathrm{rel}\ SV).$$

If we replace $S_G^{-\infty}(M; P)$ by $S_G^{-\infty}(M \times P)$, then we generally will not obtain equivalences.

Combining (43) and (44) together, we obtain the maps

$$Wh_G^{top}(M) \overset{\times P}{\to} Wh_G^{top}(M; P) \overset{\mathrm{incl}}{\leftarrow} Wh_G^{top}(M \times DV, \mathrm{rel}\ SV) \tag{45}$$

induced by (30) on the topological K-theoretical obstructions, compatible with the stable periodicity. Therefore the problem boils down to proving that the truncation

$$Wh_G^{top,\leq 0}(M) \overset{\times P}{\to} Wh_G^{top,\leq 0}(M; P) \overset{\mathrm{incl}}{\leftarrow} Wh_G^{top,\leq 0}(M \times DV, \mathrm{rel}\ SV) \tag{46}$$

of (45) induces equivalences on Tate cohomologies.

Here arises the second intricacy. The maps in (43) are not equivalences themselves. It is not hard to show that the maps in (43) indeed induce equivalences on Tate cohomologies. However, this does not imply that, after the truncation, (46) still induces equivalences on Tate cohomologies.

We get around the difficulty by repeating the proof of the periodicity on L_G for the second time.

We will often make use of the following trick: Suppose that $K_1 \to K_2 \to K_3$ is a natural fibration of involutive functors, such that there is a (not necessarily involutive) splitting $K_3 \to K_2$. Then we may take the fibres of the assemblies of homologies of the functors to obtain a new fibration of involutive functors $W_1 \to W_2 \to W_3$ with a (still not necessarily involutive) splitting $W_3 \to W_2$. The splitting enables us to truncate and still obtain a fibration of involutive functors: $W_1^{\leq 0} \to W_2^{\leq 0} \to W_3^{\leq 0}$. Moreover, since the Tate cohomology converts involutive fibrations to fibrations, we see that $\hat{H}(\mathbf{Z}_2; W_1^{\leq 0}) \to \hat{H}(\mathbf{Z}_2; W_2^{\leq 0}) \to \hat{H}(\mathbf{Z}_2; W_3^{\leq 0})$ is still a fibration. However, because the splittings was not involutive, there may no longer be splitting $\hat{H}(\mathbf{Z}_2; W_3^{\leq 0}) \to \hat{H}(\mathbf{Z}_2; W_2^{\leq 0})$ to this fibration.

In particular, if we apply our trick to (36), then we see that the functor $\hat{H}(\mathbf{Z}_2; Wh_G^{top,\leq 0}(M; ?))$ satisfies the second condition (decomposition along singularities) in the remark after the proposition 1.2. As in the proof of the periodicity of $L_G^{-\infty}$, we will check whether the functor $\hat{H}(\mathbf{Z}_2; Wh_G^{top,\leq 0}(M; ?))$ also satisfies the first condition ($\pi - \pi$ theorem) in the remark.

Thus we consider a G-manifold pair $(X, \partial X)$ satisfying isovariant $\pi - \pi$ condition. We have decomposition of functors

$$K_G^{\leq 1}(? \times (X, \partial X)) = K_G^{\leq 1}(? \times (X, \mathrm{rel}\ \partial X)) \times K_G^{\leq 1}(? \times \partial X) \tag{47}$$

with the involution described by (38), in which i_* satisfies (39). By the proposition 1.4, $M \times (X, \partial X)$ still satisfies the isovariant $\pi - \pi$ condition for any G-manifold M. Therefore i_* is an equivalence.

We take the fibre of the assembly of (47) over M/G and then truncate to obtain a decomposition

$$Wh_G^{top,\leq 0}(M; X, Y) = Wh_G^{top,\leq 0}(M; X, \mathrm{rel}\ Y) \times Wh_G^{top,\leq 0}(M; Y). \tag{48}$$

The description about the involution still holds, and i_* is still an equivalence. Such description implies that the Tate cohomology of (48) is trivial. As in the proof of the periodicity on $L_G^{-\infty}$, we see that the equivariant $\pi - \pi$ theorem is valid for $\hat{H}(\mathbf{Z}_2; Wh_G^{top,\leq 0}(M; ?))$. Consequently, we are able to show that the inclusion in (46) induces equivalence on the Tate cohomologies.

The next step is to prove that $\times P$ in (46) induces equivalence on the Tate cohomologies. In proving the periodicity of $L_G^{-\infty}$, we first proved the free action case. The corresponding case here is $\hat{H}(\mathbf{Z}_2; Wh_G^{top,\leq 0}(M, \mathrm{rel}\ M_s; ?))$, where

$$M_s = \cup_{\{1\} \neq H \subset G} M^H$$

is the nonfree part of M, so that G acts freely on $M - M_s$.

Consider the commutative diagram of functors similar to (32)

$$\begin{array}{ccc} K_G^{\leq 1}((?-?_s) \times Q) & \xleftarrow{\text{incl}} & K_G^{\leq 1}((?-?_s) \times DV, \mathrm{rel}\ SV) \\ \times_Q \uparrow & \nwarrow \phi & \downarrow \text{incl} \\ K_G^{\leq 1}(?-?_s) & \xrightarrow{\times P} & K_G^{\leq 1}((?-?_s) \times P) \end{array} \tag{49}$$

In proving the periodicity on $L_G^{-\infty}$, we have shown that $\times Q$ is an equivalence after localizing at 2, and the horizontal inclusion is an equivalence.

After taking the fibres of the assemblies over M/G of the functors in (49) and then truncating, we obtain

$$\begin{array}{ccc} & & Wh_G^{top,\leq 0}((M,\mathrm{rel}\ M_s)\times(DV,\mathrm{rel}\ SV)) \\ & & \simeq\downarrow (44) \\ Wh_G^{top,\leq 0}(M,\mathrm{rel}\ M_s;Q) & \xleftarrow{\mathrm{incl}} & Wh_G^{top,\leq 0}(M,\mathrm{rel}\ M_s;DV,\mathrm{rel}\ SV) \\ \times_Q\uparrow & \nwarrow\phi & \downarrow \mathrm{incl} \\ Wh_G^{top,\leq 0}(M,\mathrm{rel}\ M_s) & \xrightarrow{\times P} & Wh_G^{top,\leq 0}(M,\mathrm{rel}\ M_s;P) \end{array} \tag{50}$$

The map $\times Q$ is still an equivalence after localizing at 2, and the horizontal inclusion is also still an equivalence. Moreover, we have proved that the vertical inclusion induces an equivalence on the Tate cohomologies. These imply that $\times P$ induces an equivalence on the Tate cohomologies.

To prove that $\times P$ induces an equivalence on $\hat{H}(\mathbf{Z}_2; Wh_G^{top,\leq 0}(M;?))$ (not relative to the nonfree part), we consider the more general problem of the equivalence on the functor $\hat{H}(\mathbf{Z}_2; Wh_G^{top,\leq 0}(M,\mathrm{rel}\ M^{\mathcal{H}};?))$, for a collection $\mathcal{H}$ of subgroups satisfying

$$g^{-1}Kg \supset H \in \mathcal{H} \Longrightarrow K \in \mathcal{H}.$$

In case $\mathcal{H}$ is all but the trivial subgroups, we have $M^{\mathcal{H}} = M_s$, and the equivalence has been proved. The case $\mathcal{H} = \emptyset$ is what we want at the end.

Let $K \subset G$ be a maximal subgroup not in $\mathcal{H}$. Let $\mathcal{K} = \mathcal{H} \cup \{g^{-1}Kg\}$. Then we compare the theory $K^{\leq 1}$ for $?-?^{\mathcal{H}}$ and $?-?^{\mathcal{K}}$ through a diagram similar to (34)

$$\begin{array}{ccccc} K_G^{\leq 1}(?-?^{\mathcal{K}}) & \longrightarrow & K_G^{\leq 1}(?\times P-(?\times P)^{\mathcal{K}}) & \xleftarrow{\mathrm{incl}} & K_G^{\leq 1}(?\times DV-(?\times DV)^{\mathcal{K}},\mathrm{rel}\ SV) \\ \downarrow & & \downarrow & & \downarrow \\ K_G^{\leq 1}(?-?^{\mathcal{H}}) & \xrightarrow{\times P} & K_G^{\leq 1}((?-?^{\mathcal{H}})\times P) & \xleftarrow{\mathrm{incl}} & K_G^{\leq 1}((?-?^{\mathcal{H}})\times DV,\mathrm{rel}\ SV) \\ \downarrow & & \downarrow & & \downarrow \\ K_{WK}^{\leq 1}(?^K-(?^K)_s) & \xrightarrow{\times P^K} & K_{WK}^{\leq 1}((?^K-(?^K)_s)\times P^K) & \xleftarrow{\mathrm{incl}} & K_{WK}^{\leq 1}((?^K-(?^K)_s)\times DV^K,\mathrm{rel}\ SV) \end{array} \tag{51}$$

where the columns are involutive fibrations with (noninvolutive splittings). We also have the comparison similar to (35)

$$\begin{array}{ccccc} K_G^{\leq 1}(?-?^{\mathcal{K}}) & \xrightarrow{\times P} & K_G^{\leq 1}((?-?^{\mathcal{K}})\times P) & \xleftarrow{\mathrm{incl}} & K_G^{\leq 1}((?-?^{\mathcal{K}})\times DV,\mathrm{rel}\ SV) \\ \| & & \downarrow \mathrm{incl} & & \downarrow \mathrm{incl} \\ K_G^{\leq 1}(?-?^{\mathcal{K}}) & \longrightarrow & K_G^{\leq 1}(?\times P-(?\times P)^{\mathcal{K}}) & \xleftarrow{\mathrm{incl}} & K_G^{\leq 1}(?\times DV-(?\times DV)^{\mathcal{K}},\mathrm{rel}\ SV) \end{array} \tag{52}$$

If the functors in (52) are applied to the open pieces in M, then the same isotropy everywhere condition, the codimension ≥ 3 condition, and the isovariant $\pi-\pi$ structure in P imply that the all inclusions in (52) are equivalences.

Now we replace the top row of (51) with the top row of (52), take the fibres of the assemblies over M/G of the functors in the resulting diagram, and then truncate. The trick described before produces a commutative diagram

$$\begin{array}{ccccc} Wh_G^{top,\leq 0}(M,\mathrm{rel}\ M^{\mathcal{K}}) & \longrightarrow & Wh_G^{top,\leq 0}(M,\mathrm{rel}\ M^{\mathcal{K}};P) & \longleftarrow & Wh_G^{top,\leq 0}(M,\mathrm{rel}\ M^{\mathcal{K}};DV,\mathrm{rel}\ SV) \\ \downarrow & & \downarrow & & \downarrow \\ Wh_G^{top,\leq 0}(M,\mathrm{rel}\ M^{\mathcal{H}}) & \longrightarrow & Wh_G^{top,\leq 0}(M,\mathrm{rel}\ M^{\mathcal{H}};P) & \longleftarrow & Wh_G^{top,\leq 0}(M,\mathrm{rel}\ M^{\mathcal{H}};DV,\mathrm{rel}\ SV) \\ \downarrow & & \downarrow & & \downarrow \\ Wh_{WK}^{top,\leq 0}(M^K,(M^K)_s) & \longrightarrow & Wh_{WK}^{top,\leq 0}(M^K,(M^K)_s;P^K) & \longleftarrow & Wh_{WK}^{top,\leq 0}(M^K,(M^K)_s;DV^K,\mathrm{rel}\ SV^K) \end{array} \tag{53}$$

in which the columns are involutive fibrations. By the lemma 1.11, P^K is a WK-periodicity space with WK-periodicity representation V^K. Therefore by what we just proved, the bottom row of (53) induces equivalences on the Tate cohomologies. In the top row of (53), there are fewer isotropy groups in $M - M^{\mathcal{K}}$ than in $M - M^{\mathcal{H}}$. Therefore we may assume by induction that it also induces equivalences on the Tate cohomologies. Consequently, the middle row induces equivalences on the Tate cohomologies.

This completes the proof of the periodicity on the unstable structure S_G.

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