Research of Min Yan

Combinatorics: Eulerian Structure

[2] is single author. Others are joint with B. F. Chen.

1. Linear Conditions on the Number of Faces of Manifolds with Boundary.
   Advances in Applied Mathematics. **19**(1997)144-168
2. f-Vectors of Polyhedra.
   Geometriae Dedicata. **68**(1997)21-28
3. The Geometric Cone Relations for Simplicial and Cubical Complexes.
   Discrete Mathematics. **183**(1998)39-46
4. Eulerian Stratification of Polyhedra.
   Advances in Applied Mathematics. **21**(1998)22-57
5. Singularities from Eulerian Viewpoint.
   Proceedings of the Steklov Institute of Mathematics. **221**(1998)305-319
6. Eulerian 2-Strata Spaces.
   Journal of Combinatorial Theory, Series A. **85**(1999)1-28

Eulerian poset is a basic and widely used combinatorial concept. It is the combinatorial version of manifolds and has the analogue of the Poincaré duality in the form of the Dehn-Sommerville equations (linear relations among f-vectors). The concept has always been defined in the combinatorial language.

As a topologist, my main insight is that a poset is Eulerian if and only if its underlying polyhedron is a manifold from the viewpoint of the Euler characteristic (i.e., the link of any point has the same Euler number as the sphere of the right dimension). The condition is purely topological and independent of the triangulation. The insight leads to the application of the usual topological approach to the combinatorial concept.

Naturally, we may define the singularities of a polyhedron as those points whose links have the "wrong" Euler number. In [2], I showed that the number of rational linear relations among the f-vectors is $\frac{n-s-1}{2}$, where n and s are the dimensions of the polyhedron and its Eulerian singular part. This illustrates there is less "Poincaré duality" for more singular space.

In [4], we applied the stratified technique familiar to the topologists. We defined *Eulerian stratified space*, introduced the weighted combinations of f-vectors of strata, and developed the full Dehn-Sommerville theory. The theory provides a general framework for treating f-vectors for non-Eulerian posets. We also interpreted the theory as a boundary operator on the space of all weights, and computed the homology of the simplest case in [1].

The stratified consideration leads to the torsion linear relations between f-vectors that were not present in the classical combinatorial approach. In [6], we completely understand this for the case of two strata. The general case should be related to the homology introduced in [4].

[1] is our theory for the simplest manifold case. [3] is the interpretation of our result in terms of geometric cone relations. [5] is a survey of our results.

Combinatorics: Spherical Tilings by Pentagons

1. Spherical Tiling by 12 Congruent Pentagons.
   Journal of Combinatorial Theory, Series A. **120**(2013)744-776 (with H. H. Gao, N. Shi)

2. Combinatorial Tilings of the Sphere by Pentagons.
   Electronic Journal of Combinatorics. **20**(2013) #P54

3. Tilings of the Sphere by Edge Congruent Pentagons.
   arXiv:1301.0677 (with K. Y. Cheuk, H. M. Cheung)

4. Tilings of the Sphere by Geometrically Congruent Pentagons I.
   preprint. (with K. Y. Cheuk, H. M. Cheung)

5. Angle Combinations in Spherical Tilings by Congruent Pentagons.
   arXiv:1308.4207. (with H. P. Luk)

Sommerville started the classification of edge-to-edge tilings of the sphere by (geometrically) congruent triangles in 1922. The classification was completed only in 2002. We believe the classification of spherical tilings by congruent pentagons should be relatively easier (although by no means as easy as triangles) than the quadrilaterals because 5 is the other extreme among $3, 4, 5$, the numbers of sides allowed for spherical tilings.

The minimal case is the tiling by 12 congruent pentagons, which we completely classified in [1]. We outlined the strategy for the classifying in general and also classified the case that there is a tile such that all vertices have degree 3 in [4]. Now we are fairly optimistic that the complete classification can be achieved.

For triangles on the sphere, the congruence in terms of edges is the same as the congruence in terms of angles. Since this is no longer true for pentagons, we need to do separate classifications for the combinatorial, the edge length (local and global), as well as the angle (local and global) aspects. The geometrical classification is obtained by combining these.

Due to the dominance by degree 3 vertices, I investigated the distribution of high degree (i.e., > 3) vertices in [2]. I proved that combinatorially, there cannot be just one high degree vertex, and if there are two, then the tiling must be one of five *earth map tilings*. I also showed that any high degree vertex must have another one within distance 5, and if the distance is ≥ 4 for one high degree vertex, then the tiling is also the earth map tiling.

For edge congruence, [3] develops the technique for classifying edge congruent tilings in the next simplest case. A consequence is the complete classification of edge congruent earth map tilings. The result is also used in the geometrical congruence classification in [4].

For angle congruence, [5] is a systematic way that enables us, in principal, to find the complete numerical information about angle distributions. We gave complete classification for up to three angles.

Hopf Algebra

[1] joint with Y. C. Zhu. The others joint with J. H. Lu and Y. C. Zhu.

1. Stabilizer for Hopf Algebra Actions.
   Communications in Algebra. **26**(1998)3885-3898

2. Quasi-triangular Structures on Hopf Algebra with a Positive Basis.
   In: *New Trends in Hopf Algebra Theory*, 339-356, Contemporary Mathematics **267**, AMS Press 2000 (with J.H. Lu, Y.C. Zhu)

3. On the Set-theoretical Yang-Baxter Equation.
   Duke Mathematical Journal. **104**(2000)1-18

4. On Hopf Algebras with Positive Bases.
   Journal of Algebra. **237**(2001)421-445

In 1992, Drinfeld posed the question of finding set-theoretic solutions of the Yang-Baxter equation. Since the usual way of finding solutions to the Yang-Baxter equations is through quasi-triangular structures on Hopf algebras, our idea is to simply set-theoretize the whole approach. Our another idea is that any algebraic structure obtained by linearizing the set-theoretical analogue must have a basis, such that the structure coefficients are non-negative integers and are in particular ≥ 0. The idea leads to all sorts of "positive" versions of the usual algebraic concepts.

In [4], we completely classified finite dimensional "positive" Hopf algebras over $\mathbb{C}$ as always given by the bicrossproduct constructed from the unique factorizations $G = G_+G_-$ of finite groups into two subgroups. We also proved the purely set-theoretical version of the result.

In [2], we completely classified "positive" quasi-triangular structures on positive Hopf algebras, given by compatible pairs of homomorphisms between G_+ and G_-.

In [3], we completely classified set-theoretical solutions of Yang-Baxter equations, with the construction naturally inspired by [2]. The result extends the abelian case by P. Etingof, T. Schedler and A. Soloviev.

Hopf algebras can be considered as generalization of groups. In [1], we defined the analogue of the isotropy group of group actions and proved some results that are motivated by group actions.

Integrable System: Painlevé Test

All joint with J. S. Hu, and [5] also with Y.T. Yee.

1. Singularity Analysis for Integrable Systems by Their Mirrors.
   Nonlinearity **12**(1999)1531-1543

2. The Mirror Systems of Integrable Equations.
   Studies in Applied Mathematics **104**(2000)67-90
3. Local Analyticity of Solutions in the Painlevé Test.
   In *Proceedings of the Workshop on Nonlinearity, Integrability and All That: Twenty Years after NEEDS '79*, 146-152, World Scientific 2000
4. Symplectic Structure of the Painlevé Test.
   Journal of Nonlinear Mathematical Physics **8**(2001)145-148
5. Mirror Transforms of Hamiltonian Systems.
   Physica D **152-153**(2001)110-123
6. A Link between two Fundamental Contributions of Kowalevski.
   In: *The Kowalevski Property, CRM Proceedings & Lecture Notes*, 149-156, AMS Press 2002
7. An Elementary and Direct Proof of the Painlevé Property for the Painlevé Equations I, II and IV.
   Journal d'Analyse Mathématique **91**(2003)105-121
8. Painlevé Test and the Resolution of Singularities for Integrable Equations.
   preprint, arXiv:1304.7982

A simple classical view of integrable systems is Painlvé's idea that the solutions should be "single-valued" at movable singularities. The motivation is that such solutions are global and can be used to "complete the phase space". The idea leads to the Painlevé test based on formal Laurent series solutions. The test has been the most effective way of detecting the integrability.

Although the test is widely used by the practitioners, the only systematic theoretical studies were Adler and Moerbeck's work for algebraically completely integrable Hamiltonian systems and Ercolani and Siggia's examples for the Hamiltonian systems.

In [8], for ODE systems, we defined the concept of "passing the Painlevé test" in the broadest term. Then we proved that, with virtually no additional condition, passing the Painlevé test is equivalent to the existence of a good change of variable that resolves the pole singularities of the solutions (i.e., converting Laurent series to power series) while still keeping the systems of differential equations regular. We also showed that everything can be made compatible with the Hamiltonian structure if the original system is Hamiltonian.

Our change of variable can be interpreted as the transition map used for completing the phase space. What we showed is that in the best case, such a completion is automatic.

The existence and uniqueness of solution, and Cauchy-Kowalevskaya theorem on power series solution are the few truly general theorems for differential equations. Although our result is relatively more specialized, it has the same flavor and worth further exploration.

Like the classical general results, our result is only local. In [7], we prove some global integrability properties, using our regularization to get a more manageable system for delicate estimations.

The other papers are mostly examples in various settings, including some PDEs.

Topology: Equivariant Surgery

1. The Periodicity in Stable Equivariant Surgery.
   Communications on Pure and Applied Mathematics **46**(1993)1013-1040

2. Equivariant Periodicity in Surgery for Actions of Some Nonabelian Groups. In: *Proceedings of 1993 Georgia Topology Conference*, 478-508, AMS and International Press 1997

3. Equivariant Periodicity for Abelian Group Actions.
   Advances in Geometry **1**(2001)49-70 (with S. Weinberger)

4. Equivariant Periodicity for Compact Group Actions.
   Advances in Geometry **5**(2005)363-376 (with S. Weinberger)

5. Replacement Theorems for Compact Group Actions: The 2ρ Theorem.
   Pure and Applied Mathematics Quarterly **8**(2012)53-78 (with S. Cappell, S. Weinberger)

6. Closed Aspherical Manifolds with Center.
   to appear in Journal of Topology, arXiv:1108.2321 (with S. Cappell, S. Weinberger)

7. Topological Classification of Multiaxial $U(n)$-Actions.
   submitted to Journal of the European Mathematical Society, arXiv:1108.2336 (with S. Cappell, S. Weinberger)

8. Functoriality of Isovariant Homotopy Classification.
   preprint, arXiv:0909.5005 (with S. Cappell, S. Weinberger)

My topology work is about group actions on high dimensional topological (actually homological) manifolds.

In [6], we show that in all dimensions > 7 there are closed aspherical manifolds whose fundamental groups have nontrivial center but do not possess any topological circle actions. This disproves a conjectured converse (proposed by Conner and Raymond in 1970) to a classical theorem of Borel.

In a series of works [1,2,3,4], I get more and more general extension of the 4-fold periodicity in the classical surgery theory to the equivariant (more precisely, isovariant) setting. [1] is the periodicity for the representation $\mathbb{R}^4 S$ given by a suitable finite set acted by a finite group G. [2] is for some natural representations of $O(2)$ and $SU(2)$. [3] is for the circle actions and actions by compact abelian groups. [4] is the ultimate extension for compact group actions and twice of equivairant complex vector bundles, which is probably the most general equivariant 4-foldedness.

The periodicity is closely related to the replacement and rigidity properties. In [5], we proved the replacement of the fixed point set by a simple homotopic copy, for compact group actions and twice of equivariant complex vector bundles in the normal direction. This is probably the most general replacement theorem for the fixed points of the whole group. We also studied the replacement and rigidity for some other specific group representations.

In [7], we put our knowledge about the periodicity and replacement into the study of multiaxial $U(n)$-manifolds (and $Sp(n)$-manifolds). Besides M. Davis and W.C. Hsiang's earlier work in late 1970s under more restrictive context, this is the only work for the classification of manifolds acted by non-abelian, non-discrete compact groups. We showed that the topological classification can be decomposed into simpler structure sets. Using the decomposition, we get homotopy replacement for the fixed sets of half of isotropy subgroups (the case is not covered by [5]). We also explicitly computed the case of representation sphere, which greatly extends the classical result for complex projective spaces. We also showed that the suspension map for the case of representation sphere is injective.

In [8], we proved that the topological classification of the free part of manifolds acted by finite groups is covariantly functorial. The property is rather deep because it works only for topological manifolds.

Miscellaneous

1. The Higher Order Variation Method in Differential Geometry and its Applications.
   Journal of Fudan University, Natural Science **24**(1985)453-458

2. Extension of Convex Functions.
   to appear in Journal of Convex Analysis, arXiv:1207.0944

3. Size of Union.
   preprint, arXiv:1108.2339 (with Y. Z. Ou, B. L. Wang)

[1] is based on the observation that the high order variation for curve length is easy to compute when all the lower order variations vanish. This enables me to extend Synge's result about no minimal closed geodesic from surfaces of strictly positive curvature to some surfaces of non-negative curvature.

[2] gave a comprehensive treatment of the convexity for functions on the non-convex domain, especially those functions with positive definite Hessian. The problem was requested by more analytical oriented researchers (in dynamics and differential geometry) who cannot accept the prevailing way of extending convex functions by assigning $+\infty$ value. I introduced the global, local, and minimal definitions of the convexity and gave necessary and sufficient conditions for the extension of convexity in various settings. I investigated for what kind of domain, the various convexities are equivalent (or definitely not equivalent). I also studied the extension problem for the convexity defined by positive definite Hessian, which is the most relevant to the analytically oriented problems.

[3] finds the precise bounds of the size (measure as well as counting) of the union of n sets satisfying the condition that the intersection of any k sets is empty. The measure case is an elementary case of the Boolean probability bounding problem. Yet despite lots of research in more complicated setting, which never gave precise bounds, this relatively simple case is not known. The counting case is completely new.