HKUST

MATH1012 Calculus IA

Final Examination (White Version) Name:________________________

14th Dec 2015 Student ID:________________________

12:30-15:30 Lecture Section:________________________

Directions:

- This is a closed book examination. <u>No Calculator is allowed in this examination</u>.
- DO NOT open the exam until instructed to do so.
- Turn off all phones and pagers, and remove headphones. All electronic devices should be kept in a bag away from your body.
- Write your name, ID number, and Lecture Section in the space provided above, and also in the **Multiple Choice Item Answer Sheet** provided.
- DO NOT use any of your own scratch paper, and do not take any scratch paper away from the examination venue.
- When instructed to open the exam, please check that you have **11** pages of questions in addition to the cover page. Two blank pages attached at the end can be used as scratch paper.
- Answer all questions. Show an appropriate amount of work for each short or long problem. If you do not show enough work, you will get only partial credit.
- You may write on the backside of the pages, but if you use the backside, clearly indicate that you have done so.
- **Cheating is a serious violation of the HKUST Academic Code. Students caught cheating will receive a zero score for the examination, and will also be subjected to further penalties imposed by the University.**

Please read the following statement and sign your signature.

I have neither given nor received any unauthorized aid during this examination. The answers submitted are my own work.

I understand that sanctions will be imposed, if I am found to have violated the University's regulations governing academic integrity.

Student's Signature :

____

Question No.	Points	Out of
Q. 1		1
Q. 2-15		42
Q. 16		9
Q. 17		9
Q. 18		13
Q. 19		13
Q. 20		13
Total Points		100

Some useful formula:

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

Part I: Answer all of the following multiple choice questions.

- Mark your MC answers to the boxes in the Multiple Choice Item Answer Sheet provided.
- Do not forget to write your name and mark your student ID number on the Multiple Choice Item Answer Sheet.
- Mark only one answer for each MC question. Multiple answers entered for each single MC question will result in a 3 point deduction.

Write also your MC question answers in the following boxes for back up use only. The grading will be based on the answers you mark on the MC item answer sheet.

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	**Total**
Answer																

Each of the MC questions except Q1 is worth 3 points. Q1 is worth 1 point. No partial credit.

1. What is the color version of your exam paper?

   (a) Green (b) Orange (c) White (d) Yellow (e) None of the previous

2. Evaluate $f'(0)$ if
$$f(\sin x) + f(x) = 2x \qquad \text{for all } x.$$

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2

solution:
Differentiating both sides of the given relation yields
$$f'(\sin x)\cos x + f'(x) = 2 \qquad \text{for all x.}$$
Evaluate both sides at 0, we have $2f'(0) = 2$. Ans: d.

3. Evaluate the fourth derivative $f^{(4)}(\frac{\pi}{4})$ if
$$f(x) = \sin x \cos x \qquad \text{for all } x.$$

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 4 (e) 8

solution:
Since $f(x) = \frac{1}{2}\sin 2x$ for all x, we see that $f^{(4)}(x) = 8\sin 2x$ for all x. In particular, $f^{(4)}(\frac{\pi}{4}) = 8$.
Ans: e.

4. Let f, g be two functions. Evaluate $(\frac{1}{f\circ g})'(0)$ if
$$\begin{array}{llll} f(0) = 1, & f'(0) = 2, & f(1) = 1, & f'(1) = 2 \\ g(0) = 1, & g'(0) = 2, & & . \end{array}$$

(a) -4 (b) -1 (c) $\frac{1}{4}$ (d) 1 (e) 4

solution:
By chain rule,
$$(\frac{1}{f\circ g})'(0) = -\frac{f'(g(0))g'(0)}{(f(g(0)))^2} = -4.$$
Ans: a.

5. Evaluate $f'(1)$ if

$$f(x) = \int_0^x (1 + |y|)dy \qquad \text{for all } x.$$

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2

solution:
By the first version of the fundamental theorem of calculus, $f'(x) = 1+|x|$ for all x. Hence $f'(1) = 2$. Ans: e.

6. Evaluate

$$\lim_{x\to+\infty} \frac{\sqrt{4x^4+1}}{e^{x^2}}$$

if it exists.

(a) 0 (b) 1 (c) 2 (d) 4 (e) does not exist

solution:
For every $x > 1$,

$$0 \le \frac{\sqrt{4x^4+1}}{e^{x^2}} \le \frac{4x^2}{e^{x^2}}$$

and since $\lim\limits_{x\to+\infty} \frac{4x^2}{e^{x^2}} = \lim\limits_{x\to+\infty} \frac{4x}{e^x} = 0$, we see that by sandwich theorem, $\lim\limits_{x\to+\infty} \frac{\sqrt{4x^2+1}}{e^{x^2}} = 0$. Ans: a.

7. Evaluate

$$\lim_{x\to 0}(1 + x^2)^{1/x}$$

if it exists.

(a) 0 (b) $1/2$ (c) 1 (d) 2 (e) does not exist

solution:
By a standard use of l'hôpital, $\lim\limits_{x\to 0} \frac{\ln(1+x^2)}{x}$ exists and is 0. Therefore, $\lim\limits_{x\to 0}(1+x^2)^{1/x}$ exists and is $e^0 = 1$. Ans: c.

8. Evaluate

$$\lim_{x\to 0}\frac{2^x-2^{-x}}{x}$$

if it exists.

(a) 0 (b) 1 (c) $\ln 2$ (d) $\ln 4$ (e) does not exist

solution:
Both x and 2^x-2^{-x} are 0 at 0, and $\lim_{x\to 0}\frac{\ln 2(2^x)+\ln 2(2^{-x})}{1}$ exists and is $2\ln 2$. So by l'hôpital, $\lim_{x\to 0}\frac{2^x-2^{-x}}{x}$ exists and is $2\ln 2$. Ans: d.

9. Evaluate

$$\int_0^1 \frac{1+e^x}{e^x}dx.$$

(a) $1-1/e$ (b) $2-1/e$ (c) 1/e (d) $1+1/e$ (e) $2+1/e$

solution:

$$\int_0^1 \frac{1+e^x}{e^x}dx = \int_0^1 (1+e^{-x})dx = x-e^{-x}|_0^1 = 2-e^{-1}.$$

Ans: b.

10. Evaluate

$$\int_0^1 2^x dx.$$

(a) $\frac{1}{\ln 2}$ (b) $\frac{2}{\ln 2}$ (c) 1 (d) $\ln 2$ (e) $2\ln 2$

solution:

$$\int_0^1 2^x dx = \int_0^1 e^{x\ln 2}dx = \frac{1}{\ln 2}2^x|_0^1 = \frac{1}{\ln 2}.$$

Ans: a.

11. Evaluate

$$\int_{-1}^{3} \frac{xdx}{|x|}.$$

(a) -2 (b) -1 (c) 0 (d) 2 (e) 4

solution:

$$\int_{-1}^{3} \frac{xdx}{|x|} = \int_{-1}^{0} \frac{xdx}{|x|} + \int_{0}^{3} \frac{xdx}{|x|} = \int_{-1}^{0} -dx + \int_{0}^{3} dx = -1 + 3 = 2.$$

Ans: d.

12. Evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{2n} \frac{1}{4n+k}$$

if it exists. (Definite integrals may be useful.)

(a) $\frac{1}{2}$ (b) 1 (c) $\ln\frac{3}{2}$ (d) $\ln 2$ (e) $2\ln 2$

solution:
Let $f(x) = 1/x$ for $x > 0$, and subdivide $[1, 3/2]$ evenly into $2n$ subintervals. Then the corresponding (right) Riemann sum is $\sum_{k=1}^{2n} \dfrac{1}{1+\frac{k}{4n}} \dfrac{1}{4n} = \sum_{k=1}^{2n} \dfrac{1}{4n+k}$. Therefore its limit as $n \to \infty$ is $\int_1^{3/2} \dfrac{dx}{x} = \ln\dfrac{3}{2}$. Ans: c.

13. A particle is moving along a straight line so that its velocity (in ms^{-1}) after t seconds is

$$\sin(\pi t + \frac{\pi}{4}).$$

What is the displacement (in m) of this particle during the first second.

(a) $-\sqrt{2}$ (b) $-\frac{\sqrt{2}}{\pi}$ (c) 0 (d) $\frac{\sqrt{2}}{\pi}$ (e) $\sqrt{2}$

solution:
The displacement of this particle during the first second is

$$\int_0^1 \sin(\pi t + \frac{\pi}{4})dt = \frac{-1}{\pi}\cos(\pi t + \frac{\pi}{4})|_0^1 = \sqrt{2}/\pi.$$

Ans: d.

14. Let

$$f(x) = \int_0^x t(t-1)(t+1)(t^2+4)dt \qquad \text{for all } x.$$

At how many points does f have a relative (local) maximum?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

solution:
$f'(x) = x(x-1)(x+1)(x^2+1)$ for all x. Hence, $f'(x) > 0$ when $1 < x$ or $-1 < x < 0$ so that f is increasing on these two intervals. $f'(x) < 0$ when $0 < x < 1$ or $x < -1$ so that f is decreasing on these two intervals. Consequently f has a relative maximum at 0 only. Ans: b.

15. Let f and g be two functions. Which of the following is always true?

(a) If both f and g have an absolute maximum at 0, so does $f+g$.

(b) If $f+g$ has an absolute maximum at 0, so do f and g.

(c) If $f+g$ has an absolute maximum at 0, then either f or g has an absolute maximum at 0.

(d) If $f+g$ has an absolute maximum at 0, then neither f nor g have an absolute maximum at 0

(e) none of the above

solution:
If $f(x) \le f(0)$ for all x and $g(x) \le g(0)$ for all x, then $(f+g)(x) \le (f+g)(0)$ for all x. Ans: a.

Part II: Answer each of the following questions. Give your reasoning for full credits

16. ([*9 pts*]) Let

$$f(x) = x^2 \sin\frac{1}{x} \qquad \text{for all } x \neq 0.$$

(a) Define $f(0)$ so that f becomes a continuous function. [4 pts]

solution:

Since

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2 \qquad \text{for all } x,$$

and $\lim_{x\to 0} x^2$ exists and is 0. By sandwich theorem, $\lim_{x\to 0} x^2 \sin\frac{1}{x}$ exists and is 0. Thus, we define $f(0) = 0$ in order that f is continuous at 0 and hence continuous everywhere.

(b) Evaluate $f'(0)$ by applying the definition of derivatives (first principle). [5 pts]

solution:

For every $h \neq 0$,

$$\frac{f(h) - f(0)}{h} = \frac{h^2 \sin\frac{1}{h}}{h} = h \sin\frac{1}{h},$$

and $\lim_{x\to 0} h \sin\frac{1}{h}$ exists and is 0, by a reason similar to that of (a). Therefore $f'(0) = 0$.

17. ([*9 pts*]) A ladder of length L meters leans on a wall. Suppose that the top end of the ladder is falling at v meters per second when it is H meters above the ground. Calculate the rate at which the bottom end of the ladder is sliding away from the corner of the wall.
solution:
Let $x(t)$ and $y(t)$ be the distance (in meters) from the foot of the wall to the top and bottom ends of the ladder respectively after t seconds. Then

$$(x(t))^2 + (y(t))^2 = L^2 \qquad \text{for all } t.$$

Differentiating both sides yields

$$2x(t)x'(t) + 2y(t)y'(t) = 0 \qquad \text{for all } t.$$

By hypothesis, there is a T such that $y(T) = H$ (so that $x(T) = \sqrt{L^2 - (y(T))^2} = \sqrt{L^2 - H^2}$) and $y'(T) = -v$. So,

$$x'(T) = -\frac{y(T)y'(T)}{x(T)} = \frac{vH}{\sqrt{L^2 - H^2}}.$$

We see that at the given moment, the bottom end of the ladder is sliding away from the foot of the wall at a speed of $\dfrac{vH}{\sqrt{L^2 - H^2}}$ meters per second.

18. ([*13 pts*]) For each $p > 1$, show that

$$(1+x)^p \geq 1 + px \qquad \text{for all } x \geq 0.$$

solution:
For each $p > 1$, define $f(x) = (1+x)^p - px$ for all $x \geq 0$. Then,

$$f'(x) = p((1+x)^{p-1} - 1) \geq 0 \qquad \text{for all } x \geq 0.$$

So f is an increasing function on $[0, +\infty)$. We see that

$$f(x) \geq f(0) \qquad \text{for all } x \geq 0.$$

This is,

$$(1+x)^p \geq 1 + px \qquad \text{for all } x \geq 0.$$

19. ([*13 pts*]) Sketch the graph of f if $f(x) = (x^2 + 3x + 1)e^{-x}$ for all x. You should label all relative extremums of f as well as the horizontal and vertical asymptotes of the graph of f in your sketch.
solution:
First of all

$$\lim_{x \to +\infty} f(x) = 0 \text{ and } \lim_{x \to -\infty} f(x) = +\infty.$$

$$f'(x) = (-x^2 - x + 2)e^{-x} = -(x+2)(x-1)e^{-x} \begin{cases} > 0 & \text{if } -2 < x < 1 \\ < 0 & \text{if } x > 1 \text{ or } x < -2 \end{cases}$$

So f is decreasing on $[1, +\infty)$. In particular,

$$f(x) \le f(1) \qquad \text{for all } x \ge 1.$$

f is increasing on $[-2, 1]$. In particular,

$$f(-2) \le f(x) \le f(1) \qquad \text{for all } -2 \le x \le 1.$$

f is decreasing on $(-\infty, -2]$. In particular,

$$f(x) \ge f(-2) \qquad \text{for all } x \le -2.$$

Consequently, f has a relative maximum at 1 and a relative minimum at -2. Moreover,

$$\lim_{x \to +\infty} f(x) = 0 \text{ and } \lim_{x \to -\infty} f(x) = +\infty.$$

The graph of f has only one horizontal asymptote which is the x-axis, and the graph of f does not have any vertical asymptote.

20. ([*13 pts*]) Let C be the unit circle centered at the origin.

(a) For each $0 < t < \dfrac{\pi}{2}$, find the line L which is tangent to C at $(\cos t, \sin t)$. [4 pts]
solution:
The upper half of C is the graph of $f(x) = \sqrt{1-x^2}$ so that $f'(\cos t) = -\cot t$. Thus L is defined by $y - \sin t = -\cot t(x - \cos t)$, or $x \cos t + y \sin y = 1$.

(b) Where does L intersect the x-axis and the y-axis? [2 pts]
solution:
By a), L intersects the x-axis and y-axis at $(\dfrac{1}{\cos t}, 0)$ and $(0, \dfrac{1}{\sin t})$ respectively.

(c) Find the minimal length of the part of L which lies within the first quadrant. [7 pts]
solution:
The square of the length of the part of L lying in the first quadrant is

$$(\frac{1}{\cos^2 t})^2 + (\frac{1}{\sin t})^2 = \frac{1}{(\sin t \cos t)^2} = \frac{4}{\sin^2 2t} \leq \frac{4}{\sin^2 \frac{2\pi}{4}}.$$

Thus the minimal length of L is $\sqrt{4} = 2$.