

3 Dot Products

① Angle between two vectors

② Projections : How much of a vector lies in the direction of another vector.

△ Definition

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$ with $\theta \in [0, \pi]$.
 If $\vec{u} = 0$ or $\vec{v} = 0$, then $\vec{u} \cdot \vec{v} = 0$ and θ is not defined.

- a) $\theta = 0$, $\vec{u}$ and $\vec{v}$ are of the same direction
 $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}|$ with $\cos \theta = 1$
- b) $\theta = \pi$, $\vec{u}$ and $\vec{v}$ are of opposite directions.
 $\vec{u} \cdot \vec{v} = -|\vec{u}| |\vec{v}|$ with $\cos \theta = -1$
- c) $\theta = \frac{\pi}{2}$, $\cos \theta = 0$, $\vec{u} \cdot \vec{v} = 0$,
 $\vec{u}$ and $\vec{v}$ are orthogonal.

△ Two vectors are orthogonal if and only if their dot product is zero.
 Zero vector $\vec{0}$ is orthogonal to any vector.

Ex $\vec{u} = \langle 2, -6, 0 \rangle$ and $\vec{v} = \langle 0, 0, 12 \rangle$
 $\vec{u} \perp \vec{v}$, $\vec{u} \cdot \vec{v} = 0$

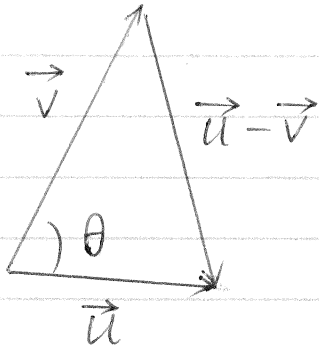
$$\vec{u} = \langle \sqrt{3}, 1 \rangle, \quad \vec{v} = \langle 0, 1 \rangle$$

$$\theta = 60^\circ \text{ or } \frac{\pi}{3}$$

$$\vec{u} \cdot \vec{v} = \sqrt{3+1} \cdot \sqrt{0+1} \cdot \cos \frac{\pi}{3} = 1$$

Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$,
 $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

Proof:



Consider the triangle formed by $\vec{u}$, $\vec{v}$, $\vec{u}-\vec{v}$.

The Law of Cosines gives

$$\begin{aligned} |\vec{u}-\vec{v}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta \\ \vec{u} \cdot \vec{v} &= |\vec{u}||\vec{v}|\cos\theta \\ &= (|\vec{u}|^2 + |\vec{v}|^2 - |\vec{u}-\vec{v}|^2)/2 \end{aligned}$$

$$|\vec{u}|^2 = u_1^2 + u_2^2 + u_3^2$$

$$|\vec{v}|^2 = v_1^2 + v_2^2 + v_3^2$$

$$|\vec{u}-\vec{v}|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2$$

$$\text{Then, } \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \quad \times$$

$$\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + u_3^2 = |\vec{u}|^2, \quad |\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

We may use this formula to evaluate θ .
 Ex. $\vec{u} = \langle \sqrt{3}, 1, 0 \rangle$, $\vec{v} = \langle 1, \sqrt{3}, 0 \rangle$
 $\vec{w} = \langle 1, \sqrt{3}, 2\sqrt{3} \rangle$

Angle between $\vec{u}$ and $\vec{v}$: $\vec{u} \cdot \vec{v} = 2\sqrt{3}$

$$|\vec{u}| = 2, \quad |\vec{v}| = 2$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}, \quad \theta = \frac{\pi}{6}$$

Angle between $\vec{u}$ and $\vec{w}$: $\vec{u} \cdot \vec{w} = 2\sqrt{3}$

$$|\vec{u}| = 2, \quad |\vec{w}| = \sqrt{4+12} = 4$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{w}}{|\vec{u}||\vec{w}|} = \frac{\sqrt{3}}{4}, \quad \theta = \cos^{-1} \frac{\sqrt{3}}{4} \approx 64.3^\circ$$

Properties:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

commutative

$$c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$$

Associative

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

distributive

Orthogonal projections 投影

Projection of $\vec{u}$ onto $\vec{v}$ is itself a vector.

$$\text{Proj}_{\vec{v}} \vec{u} = |\vec{u}| \cos \theta \frac{\vec{v}}{|\vec{v}|} \quad \text{for } \theta \in [0, \pi]$$

$$\theta \in [0, \frac{\pi}{2}) \quad \begin{array}{c} \vec{u} \\ \nearrow \theta \\ \text{Proj}_{\vec{v}} \vec{u} \rightarrow \vec{v} \end{array} : |\vec{u}| \cos \theta \frac{\vec{v}}{|\vec{v}|}$$

$$\theta \in (\frac{\pi}{2}, \pi] \quad \begin{array}{c} \vec{u} \\ \nwarrow \theta \\ \text{Proj}_{\vec{v}} \vec{u} \rightarrow \vec{v} \end{array} : -|\vec{u}| \cos \theta \left(-\frac{\vec{v}}{|\vec{v}|}\right)$$

Length · (unit vector)

$$\begin{aligned} \text{Proj}_{\vec{v}} \vec{u} &= |\vec{u}| \cos \theta \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \frac{\vec{v}}{|\vec{v}|} \\ &= \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v} \end{aligned}$$

$$\text{Proj}_{\vec{v}} \vec{u} = |\vec{u}| \cos \theta \frac{\vec{v}}{|\vec{v}|} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v}$$

$$\text{Ex } \vec{u} = \langle 4, 1 \rangle, \quad \vec{v} = \langle 3, 4 \rangle$$

$$|\vec{u}| \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{16}{5}$$

Scalar component
of $\vec{u}$ in the
direction of $\vec{v}$

$$\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

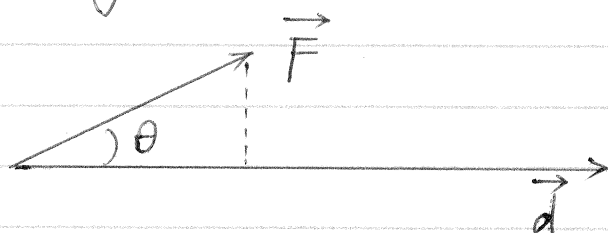
$$\text{Proj}_{\vec{v}} \vec{u} = \frac{16}{5} \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \left\langle \frac{48}{25}, \frac{64}{25} \right\rangle$$

Work and force

Definition of Work

$$W = \vec{F} \cdot \vec{d} = |\vec{F}| |\vec{d}| \cos \theta$$

A constant force $\vec{F}$ is applied to an object. The work done by $\vec{F}$ over the displacement of the object $\vec{d}$ is $W = \vec{F} \cdot \vec{d}$



$$(|\vec{F}| \cos \theta) |\vec{d}|$$

↑ Scalar component of $\vec{F}$ in the direction of $\vec{d}$.

Note that $|\vec{F}| \cos \theta$ can be positive, negative, or zero.

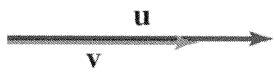
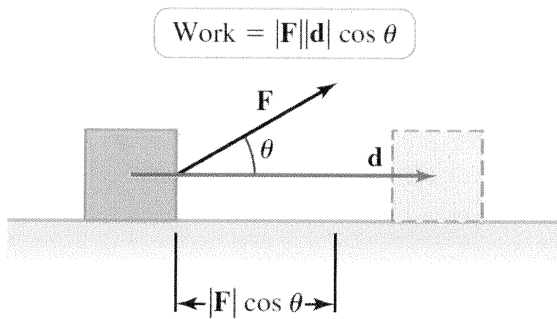
Ex. A force $\vec{F} = \langle 3, 3, 2 \rangle$ N moves an object from $P(1, 1, 0)$ to $Q(6, 6, 0)$ (in meters).

$$\vec{d} = \vec{PQ} = \langle 5, 5, 0 \rangle \text{ m}$$

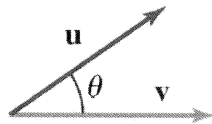
$$\vec{F} = \langle 3, 3, 2 \rangle \text{ N}$$

$$W = \vec{F} \cdot \vec{d} = 15 + 15 = 30 \text{ J}$$

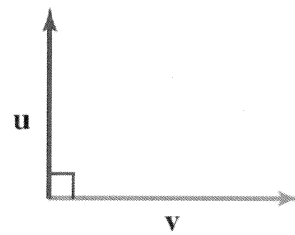
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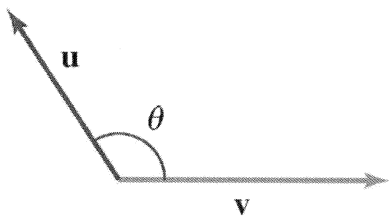
$$\theta = 0, \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$$



$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta > 0$$



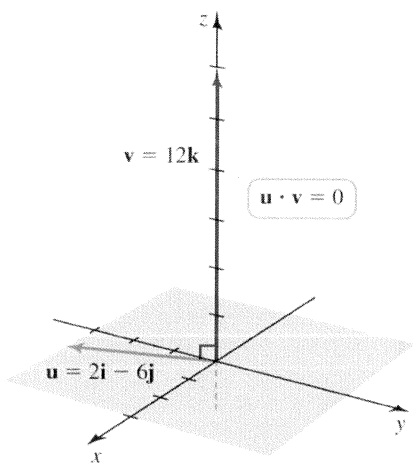
$$\theta = \frac{\pi}{2}, \mathbf{u} \cdot \mathbf{v} = 0$$



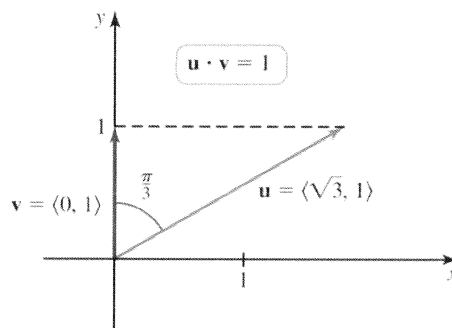
$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta < 0$$



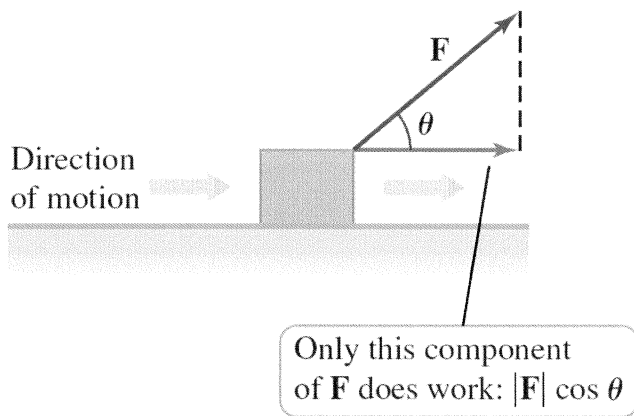
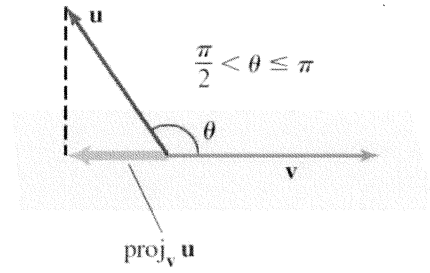
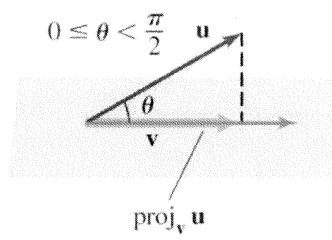
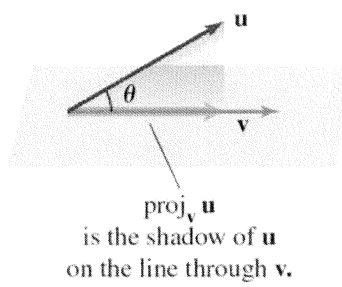
$$\theta = \pi, \mathbf{u} \cdot \mathbf{v} = -|\mathbf{u}||\mathbf{v}|$$



$$\mathbf{u} \cdot \mathbf{v} = 0$$



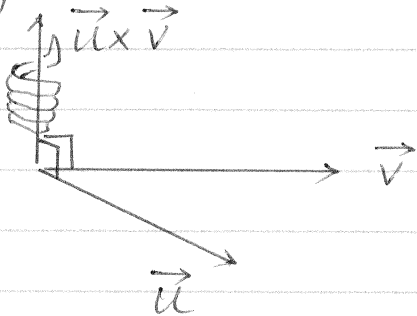
$$\mathbf{u} \cdot \mathbf{v} = 1$$



4 Cross product

Given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, the cross product $\vec{u} \times \vec{v}$ is a vector with magnitude
 $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$

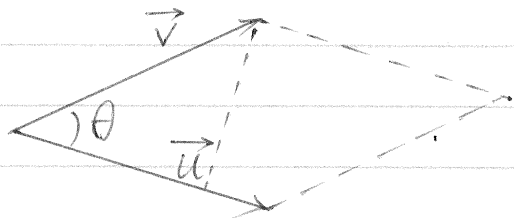
where $\theta \in [0, \pi]$ is the angle between $\vec{u}$ and $\vec{v}$.
 The DIRECTION of $\vec{u} \times \vec{v}$ is given by the right-hand rule



Let the fingers of your right hand curl from $\vec{u}$ to $\vec{v}$, the direction of $\vec{u} \times \vec{v}$ is the direction of your thumb, orthogonal to both $\vec{u}$ and $\vec{v}$.

Theorem: (1) The vectors $\vec{u}$ and $\vec{v}$ are parallel ($\theta = 0$ or π) if and only if
 $\vec{u} \times \vec{v} = \vec{0}$

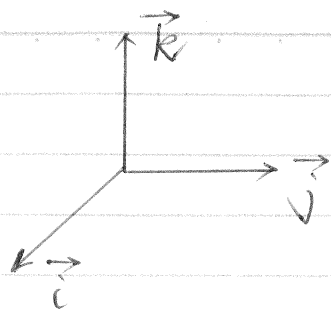
(2) If $\vec{u}$ and $\vec{v}$ are two sides of a parallelogram, then its area is
 $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta \geq 0$



$$\begin{aligned} \text{area} &= \text{base} \times \text{height} \\ &= |\vec{u}| |\vec{v}| \sin \theta \end{aligned}$$

Properties:

$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$	Anticommutative
$(a\vec{u}) \times (b\vec{v}) = ab(\vec{u} \times \vec{v})$	associative
$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$	distributive
$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$	distributive.



Cross products of
coordinate unit vectors
 $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$$

$$\begin{aligned} \vec{U} \times \vec{V} &= (u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}) \times (v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}) \\ &= u_1 v_1 \vec{0} + u_1 v_2 \vec{k} + u_1 v_3 (-\vec{j}) \\ &\quad + u_2 v_1 (-\vec{k}) + u_2 v_2 \vec{0} + u_2 v_3 \vec{i} \\ &\quad + u_3 v_1 \vec{j} + u_3 v_2 (-\vec{i}) + u_3 v_3 \vec{0} \\ &= \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \vec{k} \\ &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{aligned}$$

Ex. Area of a triangle with vertices
 $O(0, 0, 0)$, $P(2, 3, 4)$, $Q(3, 2, 0)$

$$\begin{aligned} \vec{OP} &= \langle 2, 3, 4 \rangle, \quad \vec{OQ} = \langle 3, 2, 0 \rangle \\ \vec{OP} \times \vec{OQ} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ 3 & 2 & 0 \end{vmatrix} = -8\vec{i} + 12\vec{j} - 5\vec{k} \end{aligned}$$

$$\begin{aligned} \text{Area of parallelogram} &= |-8\vec{i} + 12\vec{j} - 5\vec{k}| \\ &= \sqrt{233} \end{aligned}$$

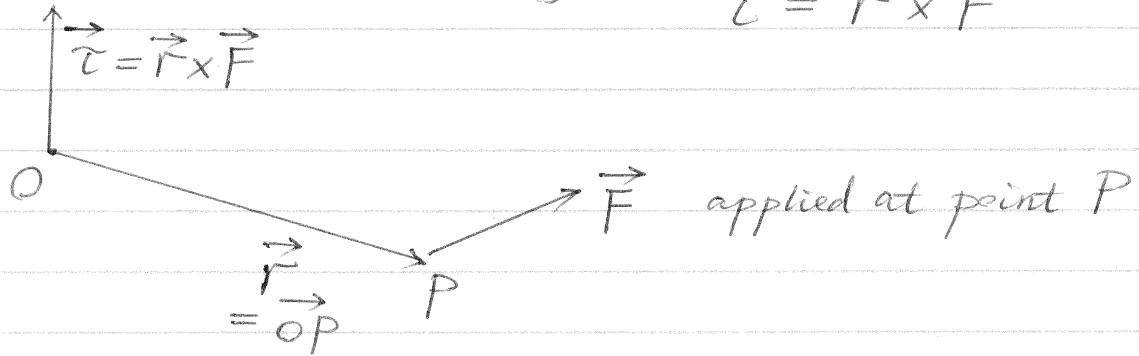
$$\text{Area of triangle} = \frac{1}{2} \sqrt{233} \quad (\text{see figure})$$

Application : Torque

A force $\vec{F}$ is applied to the point P at the head of a vector $\vec{r} = \vec{OP}$.

The torque, with twisting effect, produced by $\vec{F}$ about the point O is given by

$$\vec{\tau} = \vec{r} \times \vec{F}$$

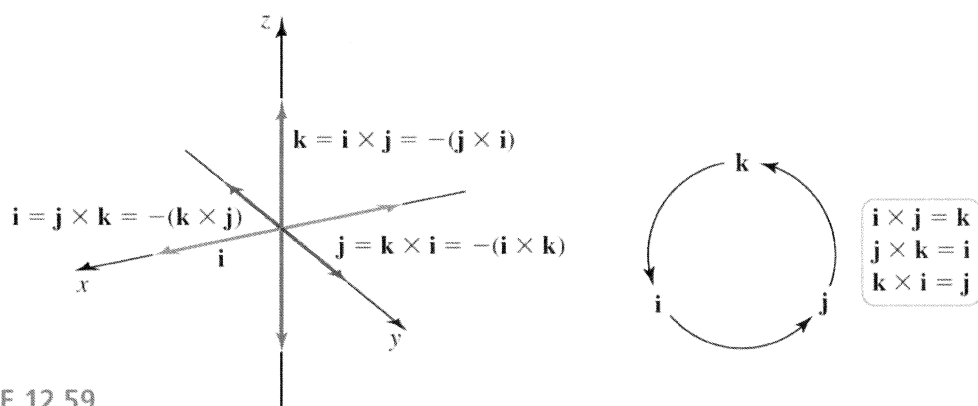
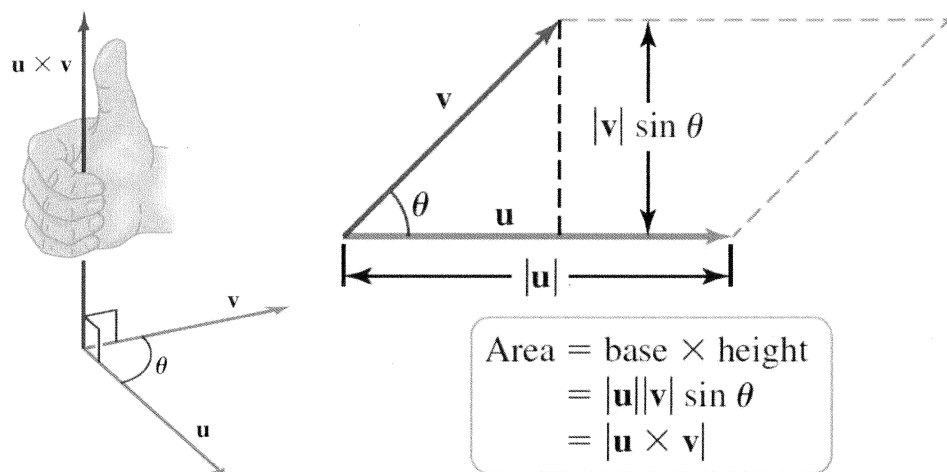


Physical explanation : direction of $\vec{\tau}$
 $\rightarrow$ direction of rotation.

$$|\vec{\tau}| = |\vec{r}| |\vec{F}| \sin \theta$$

Maximum torque occurs for $\theta = \frac{\pi}{2}$.
 $\triangle$ No torque if $\vec{r}$ and $\vec{F}$ are parallel.

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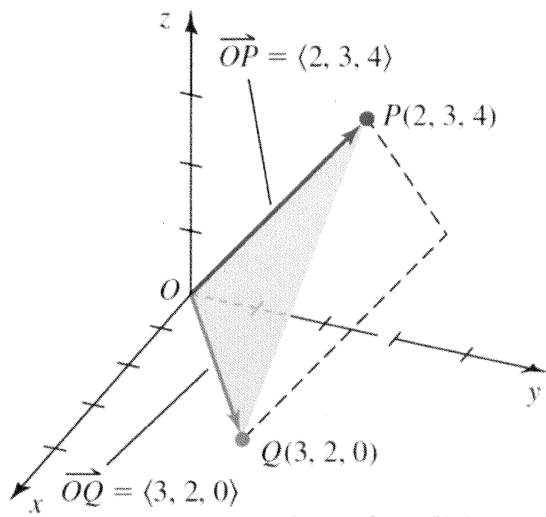


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THEOREM 12.6 Evaluating the Cross Product

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}.$$



Area of parallelogram
 $= |\vec{OP} \times \vec{OQ}|.$

Area of triangle
 $= \frac{1}{2} |\vec{OP} \times \vec{OQ}|.$

