

5. Lines and curves in space

Consider a position vector $\vec{r}$ whose components change in time.

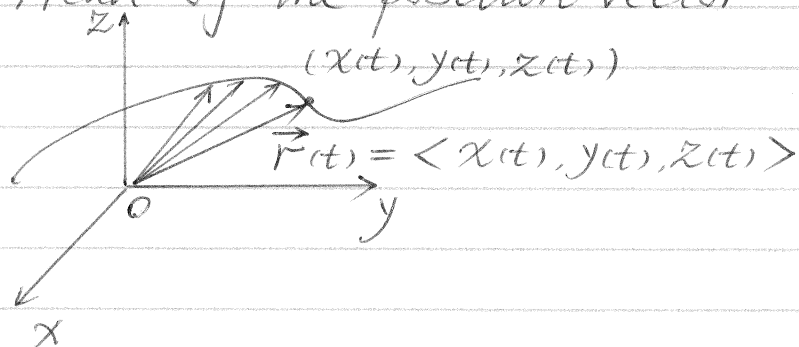
A function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ may be viewed as

1. A set of three parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$ that describe a curve in space

2. A vector-valued function: The three dependent variables x , y , and z are the components of $\vec{r}$, each varying as a function of the independent variable t .

Point $(x(t), y(t), z(t))$ in 1. $\leftrightarrow$

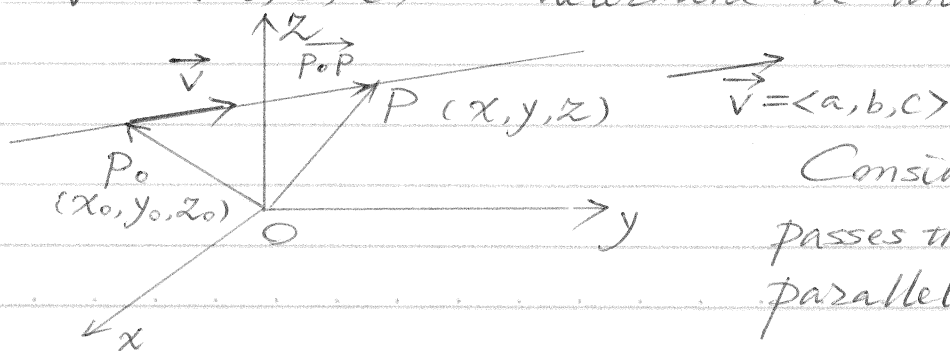
Head of the position vector $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$



A curve traced out by the point/the head as t changes.

Lines in space

One point $P_0(x_0, y_0, z_0)$ and a direction of $\vec{v} = \langle a, b, c \rangle$ determine a unique line.



Consider the line that passes through P_0 and is parallel to $\vec{v}$.

$$\vec{OP} = \vec{OP}_0 + \vec{P_0P} \quad \text{Triangle rule applied.}$$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

Here $\vec{P_0P}$ is parallel to $\vec{v}$, therefore

$$\vec{P_0P} = t\vec{v} \quad \text{where } t \text{ is a real number.}$$

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

$$\vec{r} = \langle x, y, z \rangle, \quad \vec{r}_0 = \langle x_0, y_0, z_0 \rangle \quad \text{position vectors}$$

$x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$ are the three parametric equations describing the line.

Equation of a line

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

or $\vec{r} = \vec{r}_0 + t\vec{v} = \vec{r}(t), \quad \text{for } t \in (-\infty, \infty)$

Ex 1. $P_0 (1, 2, 4)$, $\vec{v} = \langle 5, -3, 1 \rangle$

The equation for the line that passes through P_0 and is parallel to $\vec{v}$ is

$$\begin{aligned} \vec{r} = \langle x, y, z \rangle &= \langle 1, 2, 4 \rangle + t \langle 5, -3, 1 \rangle \\ &= \vec{r}_0 + t\vec{v} \end{aligned}$$

" See figures for Ex 1 and Ex 2 "

Ex 2. The line that passes through

$$P_0 (-3, 5, 8) \quad \text{and} \quad P_1 (4, 2, -1)$$

$$\vec{v} = \vec{P_0P_1} = \langle 7, -3, -9 \rangle$$

$$\begin{aligned} \vec{r} = \langle x, y, z \rangle &= \langle -3, 5, 8 \rangle + t \langle 7, -3, -9 \rangle \\ &= \vec{r}_0 + t\vec{v} \end{aligned}$$

In this example, there can be infinitely many other choices for $\vec{r}_0$ and $\vec{v}$ actually.

Projections of this line onto the xy plane and xz plane?

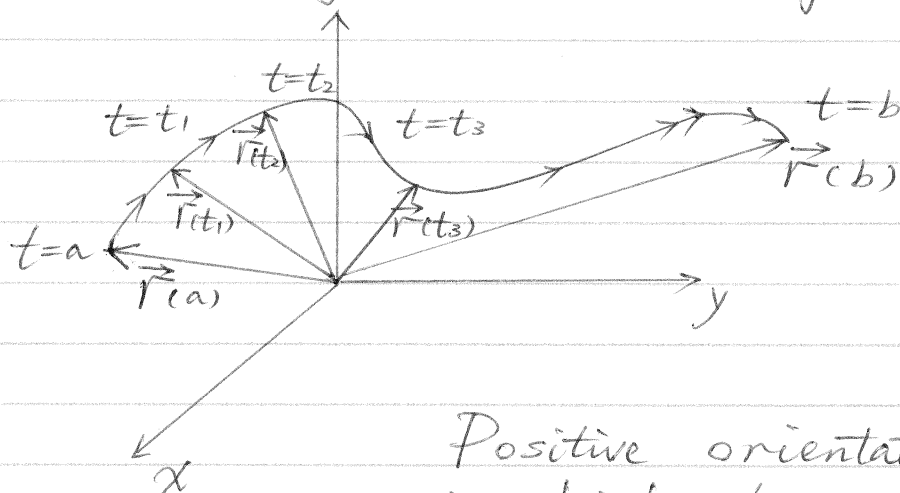
Solution: Set a particular component equal to ZERO.
 xy plane: $z = 0$.

Curves in space

Vector-valued functions of the form

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$= f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

where f , g , and h are defined for $t \in [a, b]$.As t varies over $[a, b]$, $\vec{r}(t)$ traces out a curve in space.Positive orientation: The direction in which the curve is generated as t increases from a to b .Ex A helix

"See figure"

$$\vec{r}(t) = 4\cos t \vec{i} + 5\sin t \vec{j} + \frac{t}{2\pi} \vec{k}$$

where ^(a) $t \in [0, 2\pi]$
^(b) $t \in (-\infty, \infty)$

Projection onto xy plane: $\left. \begin{array}{l} x = 4\cos t \\ y = 5\sin t \end{array} \right\} \text{An ellipse}$
 $x^2/16 + y^2/25 = 1$ $z = \frac{t}{2\pi}$: z increases from 0 to 1 as t increases from 0 to 2π . $(4, 0, 0) \rightarrow (4, 0, 1)$

Right-handed helix.

 $t \in (-\infty, \infty)$ A helix that winds around the z axis endlessly in both directions.

Ex Roller coaster curve

$$\vec{r}(t) = \cos t \vec{i} + \sin t \vec{j} + 0.4 \sin 2t \vec{k}$$

for $t \in [0, 2\pi]$

"See figure"

$x = \cos t$
 $y = \sin t$ } a circle in xy plane

$z = 0.4 \sin 2t$ varies between -0.4 and $+0.4$ with a period π . Two peaks 0.4 and two valleys -0.4 over $[0, 2\pi]$ interval.

Limits and continuity of vector-valued functions

The limit of a vector-valued function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is defined much as it is for scalar-valued functions.

Suppose

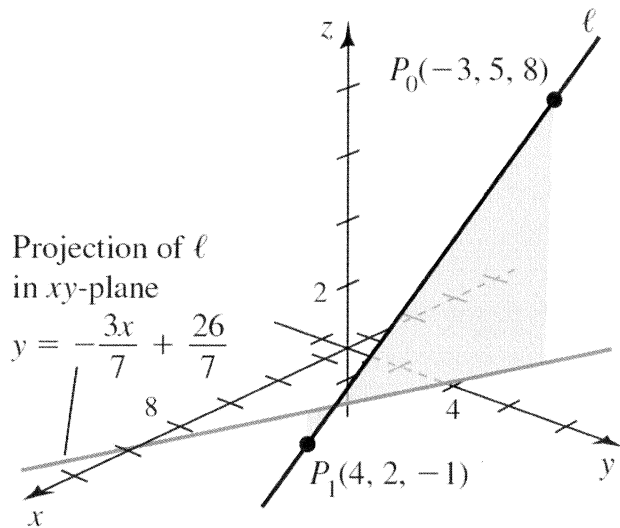
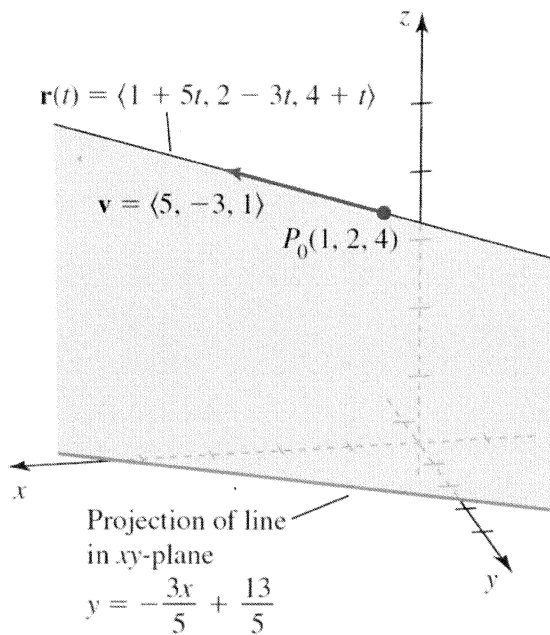
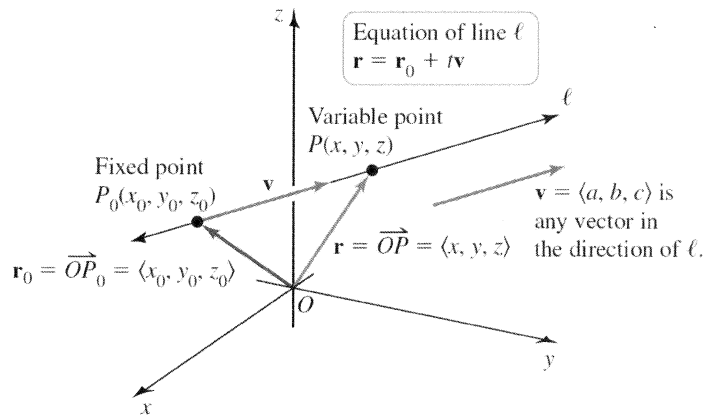
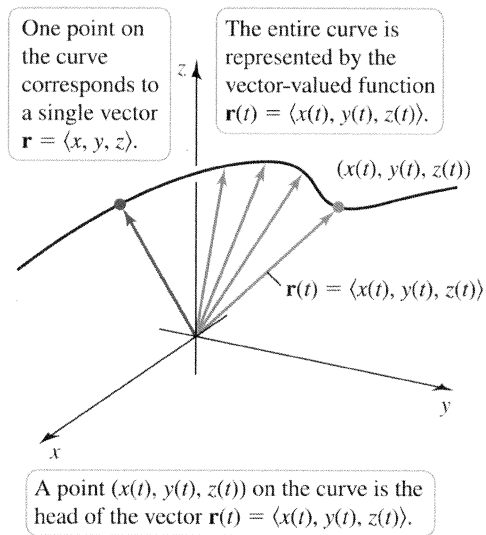
$$\lim_{t \rightarrow a} f(t) = L_1, \quad \lim_{t \rightarrow a} g(t) = L_2, \quad \lim_{t \rightarrow a} h(t) = L_3,$$

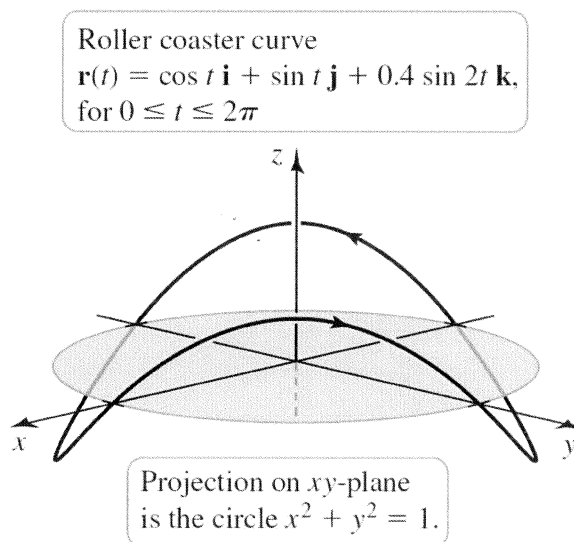
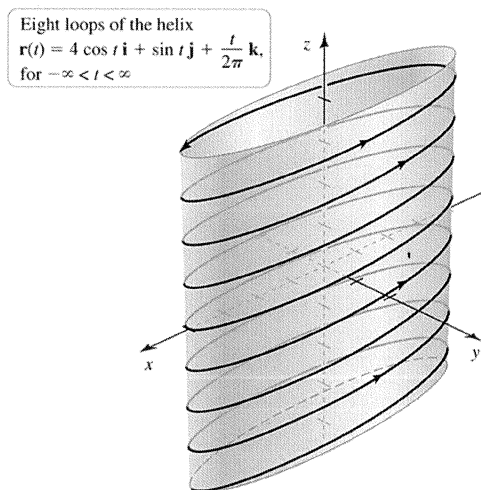
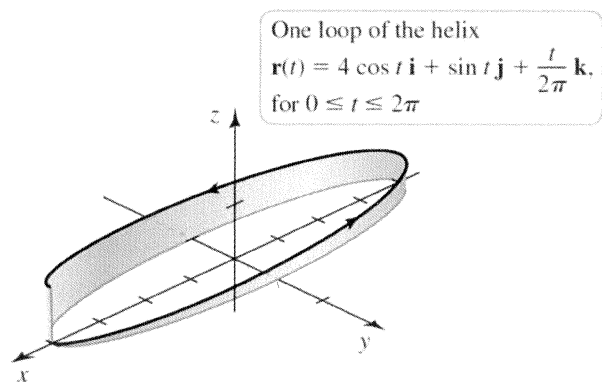
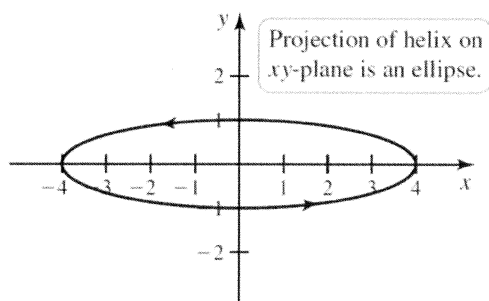
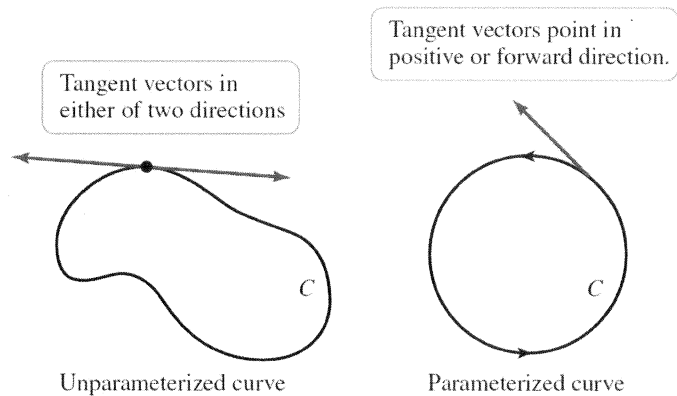
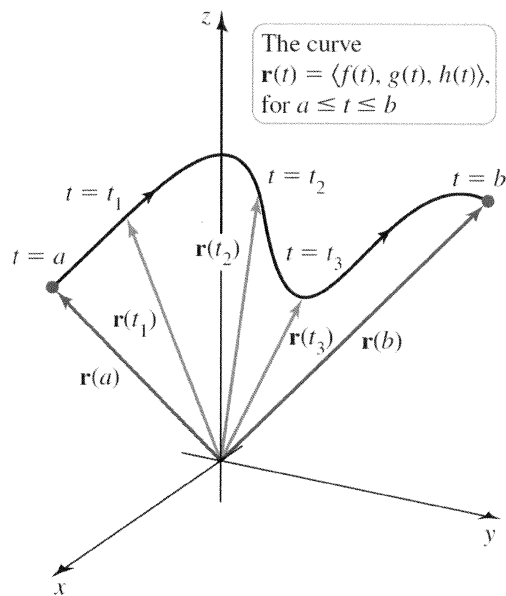
$$\begin{aligned} \text{then } \lim_{t \rightarrow a} \vec{r}(t) &= \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle \\ &= \langle L_1, L_2, L_3 \rangle \end{aligned}$$

Just computing the limits of $\vec{r}(t)$'s components

If $\vec{r}$ is a continuous function of t , then the curve it describes has no breaks or gaps, a property necessitated by the trajectory of an object.

Position must be continuous without breaks/gaps.





12.6 Calculus of vector-valued functions

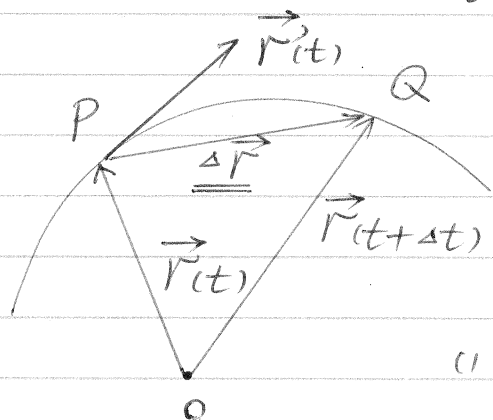
Everything you learned about differentiating and integrating functions of the form $y = f(x)$, carries over to vector-valued functions $\vec{r}(t)$.
Apply the rules for $y = f(x)$ to components of $\vec{r}(t)$.

The derivative and tangent vector

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

f, g , and h are differentiable on $t \in (a, b)$

Def:
$$\vec{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$



$$\begin{aligned} \vec{PQ} &= \Delta \vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t) \\ \vec{PQ} &= \Delta \vec{r} = \vec{OQ} - \vec{OP} \end{aligned}$$

As $\Delta t \rightarrow 0$, Q approaches P .

(1) $\vec{r}'(t)$ points in the direction of the curve. A tangent vector at P if $\vec{r}'(t) \neq \vec{0}$.

(2) Rate of change of $\vec{r}(t)$ at P (i.e., at t).

Evaluation of $\vec{r}'(t)$:

$$\begin{aligned} \vec{r}'(t) &= \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \vec{i} \\ &\quad + \lim_{\Delta t \rightarrow 0} \frac{g(t + \Delta t) - g(t)}{\Delta t} \vec{j} \\ &\quad + \lim_{\Delta t \rightarrow 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} \vec{k} \end{aligned}$$

$$= f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

If $\vec{r}'(t) \neq \vec{0}$, then $\vec{r}'(t)$ is a tangent vector in the positive direction of the curve.

Ex (a) $\vec{r}(t) = \langle t^3, 3t^2, \frac{t^3}{6} \rangle$

$$\vec{r}'(t) = \langle 3t^2, 6t, \frac{t^2}{2} \rangle \quad \text{for any } t.$$

(b)

$$\vec{r}(t) = e^{-t} \vec{i} + 10\sqrt{t} \vec{j} + 2 \cos 3t \vec{k}$$

$$\vec{r}'(t) = -e^{-t} \vec{i} + \frac{5}{\sqrt{t}} \vec{j} - 6 \sin 3t \vec{k} \quad \text{for } t > 0.$$

Unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \quad : \vec{r}'(t) \text{ normalized}$$

Ex (a) $\vec{r}(t) = \langle t^2, 4t, 4 \ln t \rangle \quad \text{for } t > 0$

$$\vec{r}'(t) = \langle 2t, 4, \frac{4}{t} \rangle$$

$$|\vec{r}'(t)| = \sqrt{4t^2 + 16 + \frac{16}{t^2}} = \sqrt{(2t + \frac{4}{t})^2} = 2t + \frac{4}{t}$$

$$\vec{T}(t) = \frac{\langle 2t, 4, \frac{4}{t} \rangle}{2t + \frac{4}{t}}$$

(b) $\vec{r}(t) = \langle 10, 3 \cos t, 3 \sin t \rangle \quad \text{for } t \in [0, 2\pi]$

$$\vec{r}'(t) = \langle 0, -3 \sin t, 3 \cos t \rangle$$

$$|\vec{r}'(t)| = 3$$

$$\vec{T}(t) = \langle 0, -\sin t, \cos t \rangle$$

Let $\vec{u}(t)$ and $\vec{v}(t)$ be differentiable vector-valued functions, let $f(t)$ be a differentiable scalar-valued function, and let $\vec{c}$ be a constant vector.

$$1. \frac{d}{dt}(\vec{c}) = \vec{0}$$

$$2. \frac{d}{dt}(\vec{u}(t) + \vec{v}(t)) = \vec{u}'(t) + \vec{v}'(t)$$

$$3. \frac{d}{dt}(f(t) \vec{u}(t)) = f'(t) \vec{u}(t) + f(t) \vec{u}'(t)$$

$$4. \frac{d}{dt}(\vec{u}(f(t))) = \vec{u}'(f(t)) f'(t)$$

$$5. \frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

$$6. \frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$$

Proof of the dot product rule

$$\frac{d}{dt}(\vec{u} \cdot \vec{v}) = \frac{d}{dt}(u_1 v_1 + u_2 v_2 + u_3 v_3)$$

$$= u_1' v_1 + u_1 v_1' + u_2' v_2 + u_2 v_2' + u_3' v_3 + u_3 v_3'$$

$$= u_1' v_1 + u_2' v_2 + u_3' v_3 + u_1 v_1' + u_2 v_2' + u_3 v_3'$$

$$= \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$$

All rules can be proved using components in a rather straightforward way.

Ex. $\vec{v}(t) = \sin t \vec{i} + 2 \cos t \vec{j} + \cos t \vec{k}$

$$(a) \frac{d}{dt}(\vec{v}(t^2)) = \underline{\vec{v}'(t^2)} (2t)$$

$$= (2t) \left(\underbrace{\cos t^2 \vec{i}}_{\frac{d}{dt} t^2} - 2 \underbrace{\sin t^2 \vec{j}}_{\vec{v}'(t^2)} - \sin t^2 \vec{k} \right)$$

$$\begin{aligned}
 (b) \quad \frac{d}{dt} (t^2 \vec{V}(t)) &= (2t) \vec{V}(t) + (t^2) \vec{V}'(t) \\
 &= 2t \langle \sin t, 2 \cos t, \cos t \rangle \\
 &\quad + t^2 \langle \cos t, -2 \sin t, -\sin t \rangle \\
 &= \langle 2t \sin t + t^2 \cos t, 4t \cos t - 2t^2 \sin t, \\
 &\quad 2t \cos t - t^2 \sin t \rangle
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad \vec{U}(t) &= t \vec{i} + t^2 \vec{j} - t^3 \vec{k} \\
 \frac{d}{dt} (\vec{U} \cdot \vec{V}) &= \vec{U}' \cdot \vec{V} + \vec{U} \cdot \vec{V}' \\
 &= \langle 1, 2t, -3t^2 \rangle \cdot \langle \sin t, 2 \cos t, \cos t \rangle \\
 &\quad + \langle t, t^2, -t^3 \rangle \cdot \langle \cos t, -2 \sin t, -\sin t \rangle \\
 &= (1 - 2t^2 + t^3) \sin t \\
 &\quad + (4t - 3t^2 + t) \cos t \\
 &= (1 - 2t^2 + t^3) \sin t + (5t - 3t^2) \cos t
 \end{aligned}$$

Integrals of vector-valued functions

An antiderivative of $\vec{r}(t)$ is $\vec{R}(t)$ if $\vec{R}'(t) = \vec{r}(t)$.

If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$,

then an antiderivative of $\vec{r}$ is

$$\vec{R}(t) = F(t)\vec{i} + G(t)\vec{j} + H(t)\vec{k}$$

where F, G, H are antiderivatives of f, g, h , respectively.

$$\vec{R}'(t) = \langle F', G', H' \rangle$$

$$= \langle f, g, h \rangle = \vec{r}(t) \quad \times$$

The indefinite integral of $\vec{r}(t)$ is

$$\int \vec{r}(t) dt = \vec{R}(t) + \vec{C}$$

where $\vec{C}$ is an arbitrary constant vector.

Ex Find $\vec{r}(t)$ with $\vec{r}(t) = \langle e^t, \sin t, t \rangle$
and $\vec{r}(0) = \vec{j}$.

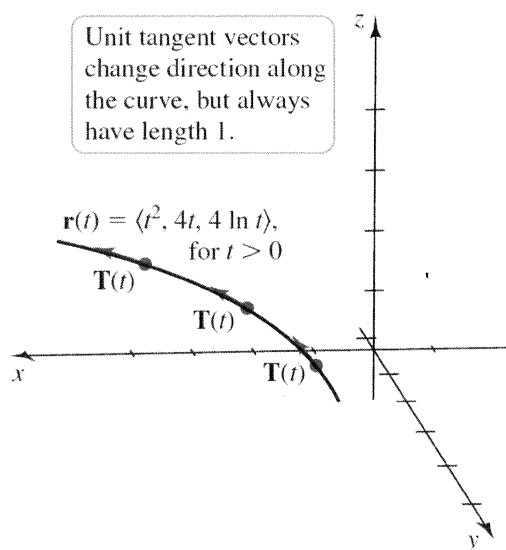
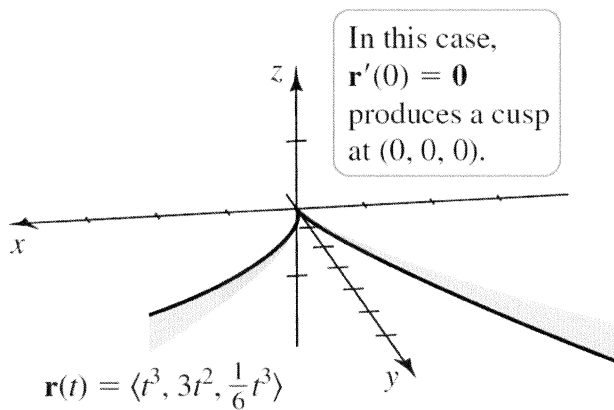
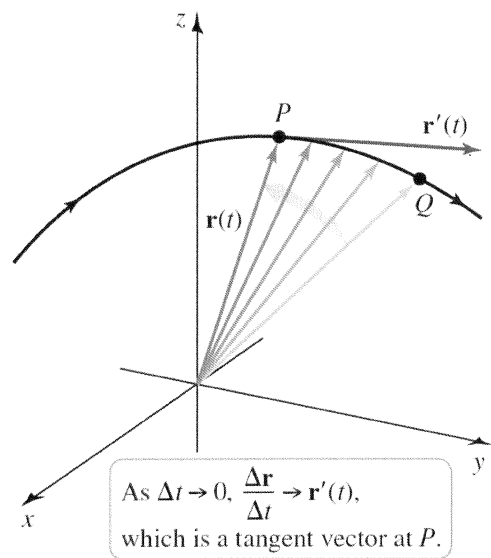
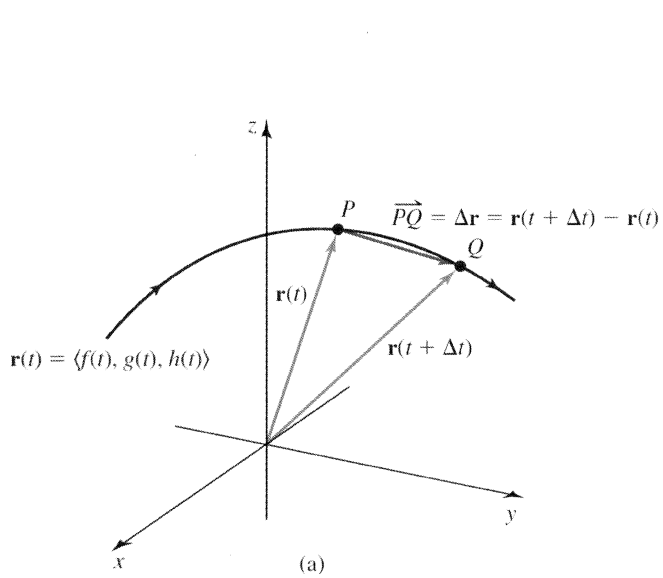
$$\begin{aligned} \Delta \quad \vec{r} &= \int \vec{r}' dt = \langle e^t, -\cos t, \frac{t^2}{2} \rangle + \underline{\underline{\vec{C}}} \\ \Delta \quad \vec{r}(0) &= \langle 0, -1, 0 \rangle + \vec{C} = \langle 0, 1, 0 \rangle = \vec{j} \\ \vec{C} &= \langle 0, 2, 0 \rangle \\ \Delta \quad \vec{r} &= \langle e^t, -\cos t + 2, \frac{t^2}{2} \rangle \end{aligned}$$

Definite integral of $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$:

$$\int_a^b \vec{r}(t) dt = \int_a^b f(t) dt \vec{i} + \int_a^b g(t) dt \vec{j} + \int_a^b h(t) dt \vec{k}$$

where f, g, h are integrable on the interval $[a, b]$

$$\begin{aligned} \text{Ex.} \quad & \int_0^{\pi} (\vec{i} + 3\cos \frac{t}{2} \vec{j} - 4t \vec{k}) dt \\ &= \int_0^{\pi} dt \vec{i} + \int_0^{\pi} 3\cos \frac{t}{2} dt \vec{j} - \int_0^{\pi} 4t dt \vec{k} \\ &= \pi \vec{i} + 6\sin \frac{t}{2} \Big|_0^{\pi} \vec{j} - 2t^2 \Big|_0^{\pi} \vec{k} \\ &= \langle \pi, 6, -2\pi^2 \rangle \end{aligned}$$



DEFINITION Unit Tangent Vector

Let $\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$