

12.7 Motion in space

Position, velocity, speed, acceleration

Consider a moving object.

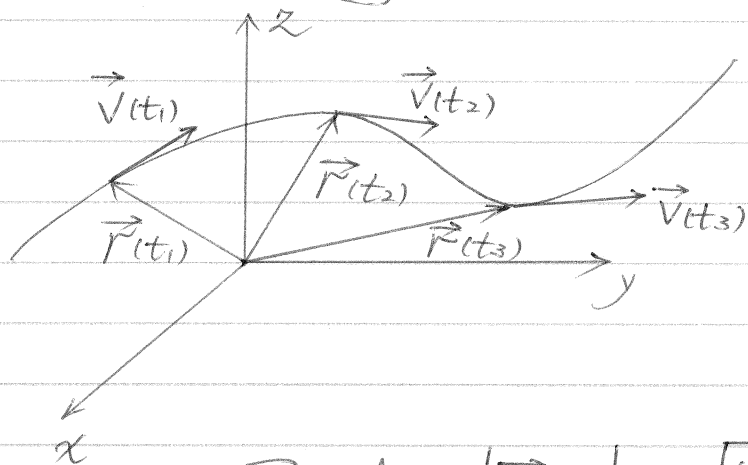
- Position $(x(t), y(t), z(t))$

Position vector

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

$\vec{r}(t)$ describes the position of a moving object for times $t \geq 0$.

The parametric curve defined by $\vec{r}(t)$ is the trajectory in space



- Velocity

$$\begin{aligned}\vec{v}(t) &= \frac{d}{dt} \vec{r}(t) \\ &= \vec{r}'(t) \\ &= \langle x'(t), y'(t), z'(t) \rangle\end{aligned}$$

Velocity is tangent to the trajectory.

Speed $|\vec{v}(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$ is a non-negative scalar-valued function.

- Acceleration $\vec{a}(t) = \frac{d}{dt} \vec{v}(t) = \frac{d^2}{dt^2} \vec{r}(t)$
 $= \vec{v}'(t) = \vec{r}''(t)$

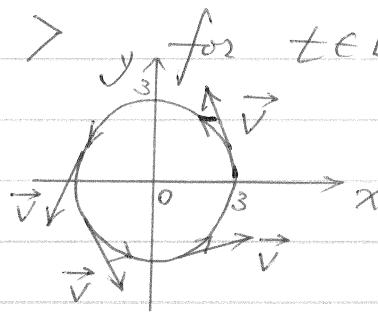
Ex $\vec{r}(t) = \langle 3 \cos t, 3 \sin t \rangle$ for $t \in [0, 2\pi]$

A sketch of the trajectory:

Velocity $\vec{v}(t) = \langle -3 \sin t, 3 \cos t \rangle$

Speed $|\vec{v}(t)| = 3$

Acceleration $\vec{a}(t) = \langle -3 \cos t, -3 \sin t \rangle$



$\vec{v} \perp \vec{r}$ and $\vec{a} \parallel \vec{r}$ at any time.

Ex $\vec{r}(t) = \langle t, t^2 - 4, \frac{t^3}{4} - 8 \rangle$ for $t \geq 0$
 $\vec{R}(t) = \langle t^2, t^4 - 4, \frac{t^6}{4} - 8 \rangle$ for $t \geq 0$
 $\vec{r}'(t) = \langle 1, 2t, \frac{3}{4}t^2 \rangle$ for velocities
 $\vec{R}'(t) = \langle 2t, 4t^3, \frac{3}{2}t^5 \rangle$

► Straight-line and circular motion

$$\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

initial position $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

constant velocity $\vec{v} = \vec{r}'(t) = \langle a, b, c \rangle$

Straight-line motion with constant velocity.

Circular motion : Motion with constant $|\vec{r}|$

Let $\vec{r}$ describes a trajectory on which $|\vec{r}|$ is constant (motion on a circle or a sphere centered at the origin). Then $\vec{r} \cdot \vec{v} = 0$, which means $\vec{r}$ and $\vec{v}$ are orthogonal at anytime (as long as $\vec{r}$ and $\vec{v}$ are defined).

Proof: $|\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t) = \text{const.}$

$$0 = \frac{d}{dt} (\vec{r}(t) \cdot \vec{r}(t)) = \vec{r}' \cdot \vec{r} + \vec{r} \cdot \vec{r}' = 2\vec{r} \cdot \vec{r}' = 2\vec{r} \cdot \vec{v}$$

Because $\vec{r} \cdot \vec{v} = 0$, $\vec{r}$ and $\vec{v}$ are orthogonal.

Ex. $\vec{r}(t) = \langle 3\cos t, 5\sin t, 4\cos t \rangle$ for $t \in [0, 2\pi]$
 $|\vec{r}(t)|^2 = 3^2 \cos^2 t + 5^2 \sin^2 t + 4^2 \cos^2 t$
 $= 25, \quad |\vec{r}(t)| = 5$

The trajectory lies on a sphere of radius 5 centered at the origin.

$$\vec{v}(t) = \langle -3\sin t, 5\cos t, -4\sin t \rangle, \quad |\vec{v}(t)| = 5$$

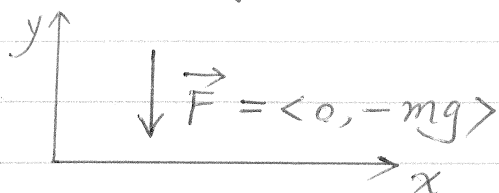
$$\vec{r}(t) \cdot \vec{v}(t) = 0, \quad \vec{r} \perp \vec{v} \text{ at anytime.}$$

Two-dimensional motion in a gravitational field.

$$\vec{F} = m\vec{a} = m\vec{r}''$$

Newton's second law of motion.

Finding the velocity and position from acceleration via integration.



$$\vec{F} = m\vec{a} = \langle 0, -mg \rangle$$

$$\vec{a} = \vec{F}/m = \langle 0, -g \rangle$$

$$g = 9.8 \text{ m/s}^2$$

gravitational acceleration

Initial position $\vec{r}(0) = \langle x_0, y_0 \rangle$

Initial velocity $\vec{v}(0) = \langle u_0, v_0 \rangle$

Two initial conditions are needed to completely determine the trajectory.

1. Solve for $\vec{v}(t)$; 2. Solve for $\vec{r}(t)$

1.

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a} \, d\tau$$

$$= \langle u_0, v_0 \rangle + \int_0^t \langle 0, -g \rangle \, d\tau$$

$$= \langle u_0, v_0 \rangle + \langle 0, -gt \rangle$$

$$= \langle u_0, v_0 - gt \rangle$$

2.

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(\tau) \, d\tau$$

$$= \langle x_0, y_0 \rangle + \int_0^t \langle u_0, v_0 - g\tau \rangle \, d\tau$$

$$= \langle x_0, y_0 \rangle + \langle u_0 t, v_0 t - \frac{1}{2} g t^2 \rangle$$

$$= \langle x_0 + u_0 t, y_0 + v_0 t - \frac{1}{2} g t^2 \rangle$$

Summary

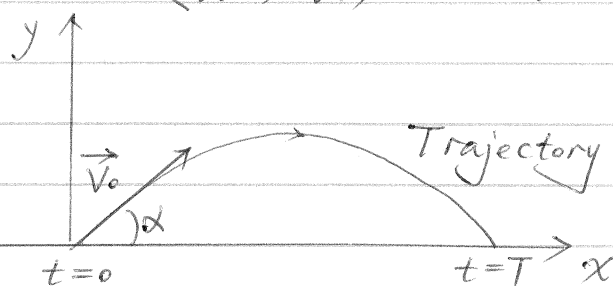
$$\underline{\vec{V}(t) = \langle u_0, v_0 - gt \rangle}$$

$$\underline{\vec{r}(t) = \langle x_0 + u_0 t, y_0 + v_0 t - \frac{1}{2}gt^2 \rangle}$$

Range, time of flight, maximum height

$$\langle u_0, v_0 \rangle = \langle |\vec{v}_0| \cos \alpha, |\vec{v}_0| \sin \alpha \rangle$$

$$\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$$



$$y(t) = v_0 t - \frac{1}{2}gt^2$$

$$y(T) = 0 \quad T \text{ is time of flight}$$

$$v_0 T - \frac{1}{2}gT^2 = 0$$

$$T = \frac{2v_0}{g} = \frac{2|\vec{v}_0| \sin \alpha}{g}$$

Range

$$x(T) = u_0 T$$

$$= |\vec{v}_0| \cos \alpha$$

$$= \frac{2|\vec{v}_0|^2 \sin \alpha \cos \alpha}{g} = \frac{|\vec{v}_0|^2 \sin 2\alpha}{g}$$

Maximum height occurs at $t = \frac{T}{2}$ The 2nd component of $\vec{v}$ vanishes at $t = \frac{T}{2}$

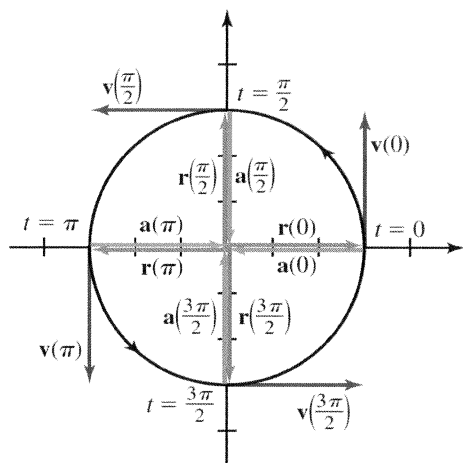
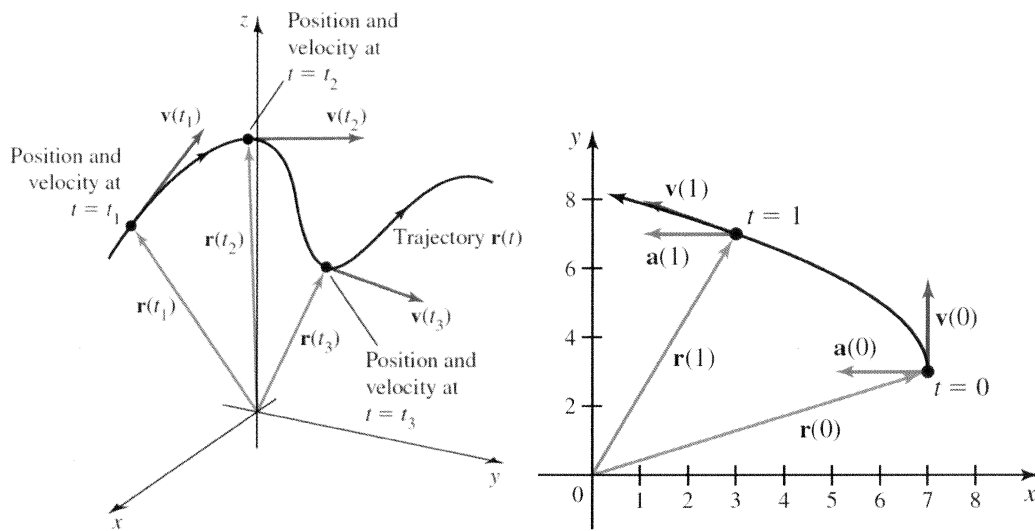
$$v_0 - g\left(\frac{T}{2}\right) = v_0 - g\frac{v_0}{g} = 0$$

$$y'\left(\frac{T}{2}\right) = 0 \rightarrow \text{maximum } y$$

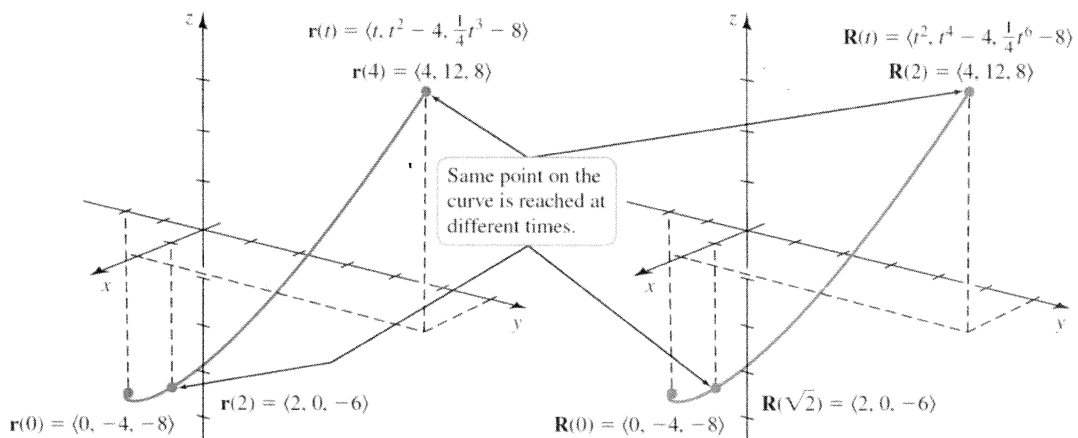
$$y\left(\frac{T}{2}\right) = v_0 \frac{T}{2} - \frac{1}{2}g\left(\frac{T}{2}\right)^2$$

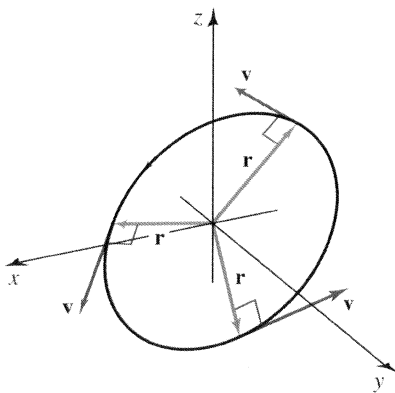
$$= v_0 \frac{v_0}{g} - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{1}{2} \frac{v_0^2}{g} = \frac{|\vec{v}_0|^2 \sin^2 \alpha}{2g}$$

12.7

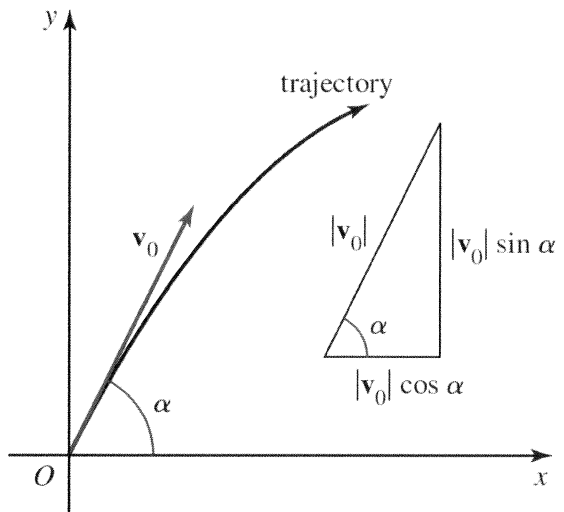
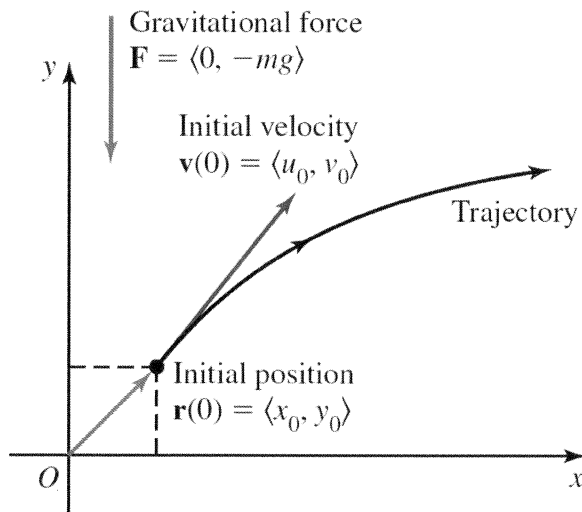
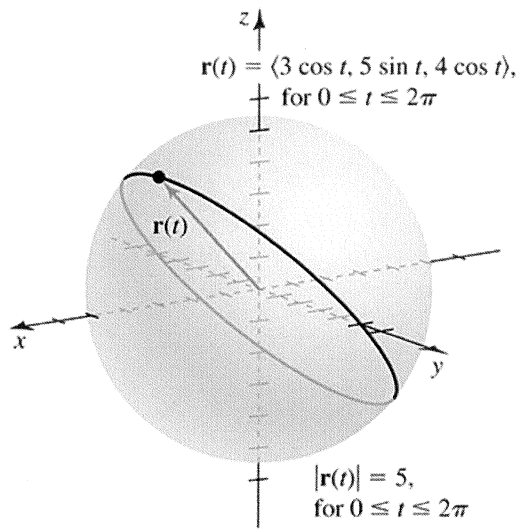


Circular motion: At all times $\mathbf{a}(t) = -\mathbf{r}(t)$ and $\mathbf{v}(t)$ is orthogonal to $\mathbf{r}(t)$ and $\mathbf{a}(t)$.



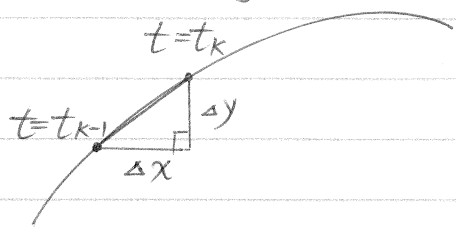


On a trajectory on which $|\mathbf{r}(t)|$ is constant, $\mathbf{v}$ is orthogonal to $\mathbf{r}$ at all points.



12.8 Length of curves

Arc length



Consider

$$\vec{r}(t) = \langle f(t), g(t) \rangle$$

- A line segment of length $\sqrt{(\Delta x)^2 + (\Delta y)^2}$

$$\Delta x = f(t_k) - f(t_{k-1})$$

$$\Delta y = g(t_k) - g(t_{k-1})$$

t_k and t_{k-1} are close enough such that Δx and Δy are small enough.

$$\Delta x \cong f'(t_k) \Delta t_k$$

$$\Delta y \cong g'(t_k) \Delta t_k$$

with $\Delta t_k = t_k - t_{k-1}$

$$\text{Then } \sqrt{(\Delta x)^2 + (\Delta y)^2} \cong \sqrt{(f'(t_k))^2 + (g'(t_k))^2} \Delta t_k$$

Summing up the length of ALL THE LINE SEGMENTS, we obtain

$$L = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{(f'(t_k))^2 + (g'(t_k))^2} \Delta t_k = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} dt$$

for the arc length of the curve over the interval $[a, b]$ for parameter t , with $b > a$.

Three dimensional curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} dt$$

$$= \int_a^b |\vec{r}'(t)| dt$$

If t is time and $\vec{r}(t)$ is the position, then $\vec{r}'(t) = \vec{v}(t)$

$$L = \int_a^b |\vec{v}(t)| dt$$

Arc length is the time integral of speed $|\vec{v}(t)|$.

Ex $\vec{r}(t) = \langle a \cos t, a \sin t \rangle$ for $t \in [0, 2\pi]$

$$\vec{r}'(t) = \langle -a \sin t, a \cos t \rangle$$

$$|\vec{r}'(t)| = a \quad (a > 0)$$

$$L = \int_0^{2\pi} |\vec{r}'(t)| dt = 2\pi a$$

Ex A helical path

$$\vec{r}(t) = \langle 250 \cos t, 250 \sin t, 100t \rangle$$

$$\text{for } t \in [0, 10]$$

$$\vec{r}'(t) = -250 \sin t \vec{i} + 250 \cos t \vec{j} + 100 \vec{k}$$

$$|\vec{r}'(t)| = \sqrt{250^2 + 100^2} \approx 269$$

$$L = \int_0^{10} |\vec{r}'(t)| dt \approx 2690$$

Arc length of polar curves $r = f(\theta)$

$$x = r \cos \theta = f(\theta) \cos \theta, \quad y = r \sin \theta = f(\theta) \sin \theta$$

$$\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta$$

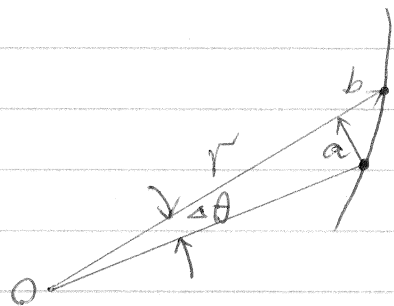
$$\frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta$$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{(f(\theta))^2 + (f'(\theta))^2}$$

Geometrical picture:

$f'(\theta) \Delta \theta$, "radial" extension

$f(\theta) \Delta \theta$, "angular" extension



$a \approx r \Delta \theta = f(\theta) \Delta \theta$, along angular direction

$b \approx \Delta r = f'(\theta) \Delta \theta$, along radial direction

a and b are nearly perpendicular.

Arc length of a polar curve $r = f(\theta)$
on the interval $\theta \in [\alpha, \beta]$:

$$L = \int_{\alpha}^{\beta} \sqrt{(f(\theta))^2 + (f'(\theta))^2} d\theta$$

A special case of parametric curves, with distinct geometrical meaning in $\sqrt{f^2 + (f')^2} d\theta$.

Ex. (a) $r = \theta$ for $\theta \in [0, 2\pi]$: A spiral
(b) $r = 1 + \cos \theta$: Cardioid

$$\begin{aligned} \text{(a)} \quad f &= \theta, \quad f' = 1 \\ L &= \int_0^{2\pi} \sqrt{f^2 + (f')^2} d\theta = \int_0^{2\pi} \sqrt{\theta^2 + 1} d\theta \\ &= \left[\frac{1}{2} \theta \sqrt{\theta^2 + 1} + \frac{1}{2} \ln(\theta + \sqrt{\theta^2 + 1}) \right] \Big|_0^{2\pi} \\ &= \pi \sqrt{4\pi^2 + 1} + \frac{1}{2} \ln(2\pi + \sqrt{4\pi^2 + 1}) \quad \# \end{aligned}$$

Check the antiderivative for $\sqrt{\theta^2 + 1}$ from one-variable calculus techniques.

$$\begin{aligned} \text{(b)} \quad f &= 1 + \cos \theta, \quad f' = -\sin \theta \\ \sqrt{f^2 + (f')^2} &= \sqrt{1 + 2\cos \theta + \cos^2 \theta + \sin^2 \theta} \\ &= \sqrt{2 + 2\cos \theta} = \sqrt{4 \cos^2 \frac{\theta}{2}} \end{aligned}$$

We calculate the arc length for $\theta \in [0, \pi]$ and then multiply it by a factor of 2.

$$\begin{aligned} L &= 2 \int_0^{\pi} \sqrt{f^2 + (f')^2} d\theta = 2 \int_0^{\pi} \sqrt{4 \cos^2 \frac{\theta}{2}} d\theta \\ &= 4 \int_0^{\pi} \cos \frac{\theta}{2} d\theta \end{aligned}$$

$$\begin{aligned} &= 8 \sin \frac{\theta}{2} \Big|_0^{\pi} = 8 \quad \# \\ \text{See figures} \\ \text{for (a) and (b)} \end{aligned}$$

▲ The arc length function

Consider a curve represented by

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \text{ for } t \geq a.$$

As t increases, the length of the curve also increases.

The length of the curve from $\vec{r}(a)$ to $\vec{r}(t)$:

$$\begin{aligned} S(t) &= \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2} \, du \\ &= \int_a^t |\vec{r}'(u)| \, du \end{aligned}$$

$S(t)$ increases with increasing t obviously.

$$\frac{d}{dt} S = \frac{d}{dt} \int_a^t |\vec{r}'(u)| \, du = |\vec{r}'(t)|$$

▲ Arc length parametrization

When $S = S(t)$ is obtained, we can use S to replace t as the parameter that parametrizes the same curve.

Ex. Consider the helix $\vec{r}(t) = \langle 2\cos t, 2\sin t, 4t \rangle$ for $t \geq 0$.

$$\begin{aligned} \sqrt{(x')^2 + (y')^2 + (z')^2} &= \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 4^2} \\ &= \sqrt{20} = 2\sqrt{5} \end{aligned}$$

$$S = S(t) = \int_0^t 2\sqrt{5} \, du = 2\sqrt{5}t$$

$$t = \frac{S}{2\sqrt{5}}, \text{ Substituted into } \vec{r}(t):$$

$$\begin{aligned} \vec{r}_s(s) &= \vec{r}\left(t = \frac{S}{2\sqrt{5}}\right) = \left\langle 2\cos \frac{S}{2\sqrt{5}}, 2\sin \frac{S}{2\sqrt{5}}, 4\frac{S}{2\sqrt{5}} \right\rangle \\ &= \left\langle 2\cos \frac{S}{2\sqrt{5}}, 2\sin \frac{S}{2\sqrt{5}}, \frac{2S}{\sqrt{5}} \right\rangle \end{aligned}$$

$$\vec{r}_s(s) = \vec{r}(t)$$

with $t = t(s)$

$$S = S(t) \rightarrow t = t(s)$$

Inverse

12.8

