

12.9 Curvature and normal vectors

▲ Unit tangent vector

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a smooth oriented curve, then the unit tangent vector at a point is

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

where $\vec{v}(t) = \vec{r}'(t)$ is a tangent vector.

How quickly does $\vec{T}$ change (in direction) along the curve?

▲ Curvature

$$K(s) = \left| \frac{d\vec{T}}{ds} \right|$$

Rate of change of $\vec{T}$ with respect to arc length $\frac{d\vec{T}}{ds}$. Its magnitude $\left| \frac{d\vec{T}}{ds} \right|$ is the curvature.

$$K = \left| \frac{d\vec{T}}{ds} \right| = \left| \frac{d\vec{T}/dt}{ds/dt} \right| = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

in which $\frac{ds}{dt} = |\vec{r}'| = |\vec{v}|$ is used.

$$K = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

Ex Lines $\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$
 $\vec{r}'(t) = \vec{v}(t) = \vec{v}$, a constant vector
 $\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle a, b, c \rangle}{\sqrt{a^2 + b^2 + c^2}}$

$\vec{T}$ is a constant vector, $\frac{d\vec{T}}{dt} = \vec{0}$

$K=0$: Zero curvature anywhere on the line.

Ex Circles $\vec{r}(t) = \langle R \cos t, R \sin t \rangle$
 $R > 0$ for $t \in [0, 2\pi]$

$$\vec{r}'(t) = \vec{v}(t) = \langle -R \sin t, R \cos t \rangle$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \langle -\sin t, \cos t \rangle, \text{ unit tangent vec.}$$

$$\frac{d\vec{T}}{dt} = \langle -\cos t, -\sin t \rangle$$

$$K = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right| = \frac{1}{R} : \text{constant curvature}$$

An alternative curvature formula

Consider a smooth curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ with $\vec{v}(t) = \vec{r}'(t)$ and $\vec{a}(t) = \vec{v}'(t)$, which are velocity and acceleration, respectively.

$$\vec{v} = |\vec{v}| \frac{\vec{v}}{|\vec{v}|} = |\vec{v}| \vec{T}$$

↑ speed unit tangent vector.

$$\vec{a} = \frac{d}{dt} \vec{v} = \frac{d}{dt} (|\vec{v}| \vec{T}) = \left(\frac{d}{dt} |\vec{v}| \right) \vec{T} + |\vec{v}| \frac{d\vec{T}}{dt}$$

[product rule]

$$\vec{v} \times \vec{a} = (|\vec{v}| \vec{T}) \times \left[\underbrace{\left(\frac{d}{dt} |\vec{v}| \right) \vec{T}}_{\text{parallel to } \vec{T} \rightarrow \text{zero vector for cross product}} + |\vec{v}| \frac{d\vec{T}}{dt} \right]$$

$$= |\vec{v}|^2 \vec{T} \times \frac{d\vec{T}}{dt}$$

$$|\vec{v} \times \vec{a}| = |\vec{v}|^2 \left| \vec{T} \times \frac{d\vec{T}}{dt} \right|$$

$$= |\vec{v}|^2 |\vec{T}| \left| \frac{d\vec{T}}{dt} \right| \sin \theta$$

θ is the angle between $\vec{T}$ and $\vec{T}'(t) = \frac{d\vec{T}}{dt}$.

Because $\vec{T} \cdot \vec{T} = 1$ for unit vector,
 $\vec{T}$ and $\frac{d\vec{T}}{dt}$ are orthogonal: $\theta = \frac{\pi}{2}$

$$\left[\frac{d}{dt}(\vec{T} \cdot \vec{T}) = \vec{T}' \cdot \vec{T} + \vec{T} \cdot \vec{T}' = 2\vec{T} \cdot \vec{T}' = 0, \quad \vec{T} \perp \vec{T}' \right]$$

$$|\vec{v} \times \vec{a}| = |\vec{v}|^2 \left| \frac{d\vec{T}}{dt} \right|$$

Since $\kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$, we have

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

✱ Alternative formulae
for curvature

Ex $\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle$
 $a > 0$ and $b > 0$: a helix.

$$\vec{v} = \langle -a \sin t, a \cos t, b \rangle = \vec{r}'(t)$$

$$\vec{a} = \langle -a \cos t, -a \sin t, 0 \rangle = \vec{v}'(t)$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a \sin t & a \cos t & b \\ -a \cos t & -a \sin t & 0 \end{vmatrix} = ab \sin t \vec{i} - ab \cos t \vec{j} + a^2 \vec{k}$$

$$|\vec{v} \times \vec{a}| = \sqrt{a^2 b^2 + a^4} = a \sqrt{a^2 + b^2}$$

$$|\vec{v}| = \sqrt{a^2 + b^2}$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{a}{a^2 + b^2}$$

$b \rightarrow 0$: Circle, $\kappa \rightarrow \frac{1}{a} = \frac{1}{\text{radius}}$

$b \rightarrow \infty$: $\kappa \rightarrow 0$ straightened

Ex $\vec{r}(t) = \langle t, at^2, 0 \rangle$, $a > 0$ a parabola.

$$\vec{v}(t) = \langle 1, 2at, 0 \rangle$$

$$\vec{a}(t) = \langle 0, 2a, 0 \rangle$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2at & 0 \\ 0 & 2a & 0 \end{vmatrix} = 2a\vec{k}$$

$$k = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{2a}{(1+4a^2t^2)^{\frac{3}{2}}}$$

Maximum curvature at $t=0$: $k_{\max} = 2a$

Fastest turn of $\vec{T}$ at the origin.

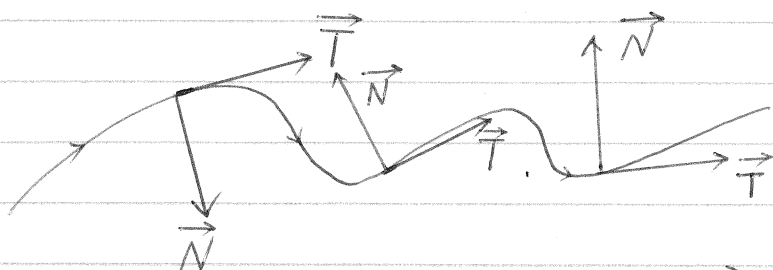
Principal Unit Normal Vector $\vec{N}$

k is the magnitude of $\frac{d\vec{T}}{ds}$: $k = \left| \frac{d\vec{T}}{ds} \right|$

How about the direction of $\frac{d\vec{T}}{ds}$?

$\vec{N}(s) = \frac{d\vec{T}/ds}{|d\vec{T}/ds|}$: unit vector in the direction of $d\vec{T}/ds$: Principal unit normal vector

$$\frac{d\vec{T}}{ds} = \left| \frac{d\vec{T}}{ds} \right| \vec{N}(s) = k(s) \vec{N}(s)$$

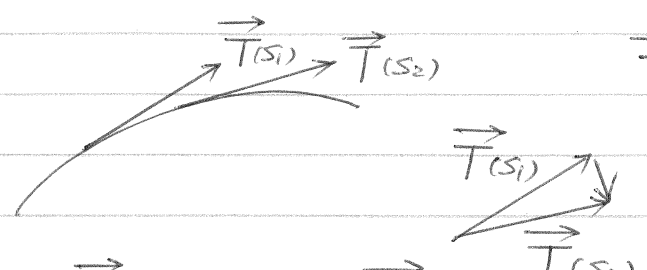


Two properties : 1. $\vec{T} \cdot \vec{N} = 0$, $\vec{T}$ and $\vec{N}$ are orthogonal

2. $\vec{N}$ points to the inside of the curve.

Proof: 1. $\vec{T} \cdot \vec{T} = 1$ for unit vectors

Therefore, $\vec{T}$ and $\frac{d\vec{T}}{ds}$ are orthogonal, and so are $\vec{T}$ and $\vec{N}$.

2.  $\vec{T}(s_2) - \vec{T}(s_1)$ for $s_2 - s_1 > 0$

$$\frac{d\vec{T}}{ds} = \lim_{s_2 - s_1 \rightarrow 0} \frac{\vec{T}(s_2) - \vec{T}(s_1)}{s_2 - s_1}$$

$\frac{d\vec{T}}{ds}$ points approximately in the direction of $\vec{T}(s_2) - \vec{T}(s_1)$ for small positive $s_2 - s_1$.

Therefore, $\vec{N}$ and $\frac{d\vec{T}}{ds}$ point to the inside, the direction that the curve is turning.

Ex. A helix $\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle$ with $a > 0$ and $b > 0$.

$$\vec{N} = \frac{d\vec{T}/ds}{|d\vec{T}/ds|} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} \quad \text{with } \frac{d\vec{T}}{ds} \text{ and } \frac{d\vec{T}}{dt} \text{ in the same direction}$$

$$\vec{v} = \langle -a \sin t, a \cos t, b \rangle$$

$$|\vec{v}| = \sqrt{a^2 + b^2}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -a \sin t, a \cos t, b \rangle}{\sqrt{a^2 + b^2}}$$

$$\frac{d\vec{T}}{dt} = \frac{\langle -a \cos t, -a \sin t, 0 \rangle}{\sqrt{a^2 + b^2}}$$

$$\frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \langle -\cos t, -\sin t, 0 \rangle = \vec{N}$$

Note that $\vec{N}$ is a unit vector parallel to the xy plane.

