

Functions of Several Variables

13.1 Planes and surfaces

$y = f(x)$: curves in $\mathbb{R}^2$

$z = f(x, y)$: surfaces in $\mathbb{R}^3$

The simplest surface in $\mathbb{R}^3$: A plane

Given a fixed point P_0 and a nonzero normal vector denoted by $\vec{n}$, the set of points P for which $\vec{P_0P}$ is orthogonal to $\vec{n}$ is called a plane.

$$\vec{n} = \langle a, b, c \rangle, \quad P_0 (x_0, y_0, z_0), \quad P (x, y, z)$$

$$\vec{n} \cdot \vec{P_0P} = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\blacktriangle a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\blacktriangle ax + by + cz = d, \quad d = ax_0 + by_0 + cz_0$$

General equation of a plane in $\mathbb{R}^3$.

Ex 1. $P_0 (2, -3, 4)$, normal vector $\vec{n} = \langle -1, 2, 3 \rangle$

$$(-1)(x - 2) + 2(y + 3) + 3(z - 4) = 0$$

$$-x + 2y + 3z = 4$$

Ex 2. A plane through three points

$P (2, -1, 3)$

$Q (1, 4, 0)$

$R (0, -1, 5)$

$$\text{Normal vector} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 5 & -3 \\ -2 & 0 & 2 \end{vmatrix} = 10\vec{i} + 8\vec{j} + 10\vec{k}$$

$$10(x - 2) + 8(y + 1) + 10(z - 3) = 0$$

$$5x + 4y + 5z = 21.$$

Parallel and orthogonal planes

- △ Two distinct planes are parallel if their normal vectors are parallel.
- △ Two distinct planes are orthogonal if their respective normal vectors are orthogonal.

Conditions for parallel vectors and orthogonal vectors

$\downarrow$
 (scalar multiple)

$\downarrow$
 (dot product)

Ex. Find the plane that passes through $(-2, 4, 1)$ and is parallel to $3x - 2y + z = 4$

$\vec{n} = \langle 3, -2, 1 \rangle$ normal vector.

$$3(x+2) - 2(y-4) + (z-1) = 0$$

$$3x - 2y + z = -13$$

Ex. The line of intersection of $x + 2y + z = 5$ and $2x + y - z = 7$

$$\vec{n}_1 = \langle 1, 2, 1 \rangle$$

$$\vec{n}_2 = \langle 2, 1, -1 \rangle$$

$\vec{n}_1$ and $\vec{n}_2$ are not parallel: The two planes intersect.

The intersection line l is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$ because l is in both planes.

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -3\vec{i} + 3\vec{j} - 3\vec{k}$$

A point on the line l : Suppose $z = 0$, $\begin{cases} x + 2y = 5 \\ 2x + y = 7 \end{cases}$

$(3, 1, 0) \rightarrow$ solution for a point on l .

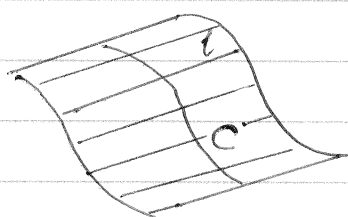
$$\vec{r}(t) = \underbrace{\langle 3, 1, 0 \rangle}_{\vec{r}_0} + t \underbrace{\langle -3, 3, -3 \rangle}_{\vec{v}}$$

Cylinders and traces

▲ Cylinder's definition

Given a curve C in a plane P and a line l not in P , a cylinder is the surface consisting of all lines parallel to l that pass through C .

See figure.



A common situation:

The curve is in a coordinate plane and l is in the direction of the 3rd coordinate axis.

Ex. The curve $y = z^2$ in the yz plane and l is parallel to x axis.

The variable corresponding to the coordinate axis parallel to l is missing: $y = z^2$ is a cylinder parallel to x axis.

▲ Trace

A trace of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes.

Traces in the coordinate planes are called the xy -trace, the xz -trace, and the yz -trace.

Graphing cylinders: (See figures)

a) $x^2 + 4y^2 = 16$, a cylinder parallel to z axis.

b) $x - \sin z = 0$, a cylinder parallel to y axis.

Quadric surfaces

The general quadratic equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

Not all of A, B, C, D, E, F are zero.

Making hand sketches can be challenging.

1. Intercepts, where the surface intersects the ^{coordinate} axes.
2. Traces, which help visualize the surface
3. Sketch.

An ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$a=3, b=4, c=5$

An elliptic paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

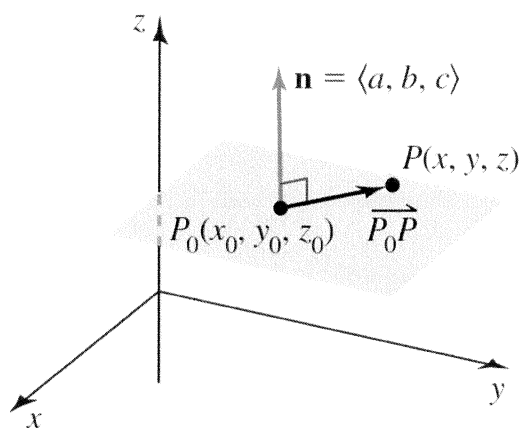
$a=4, b=2$

A hyperboloid of one sheet

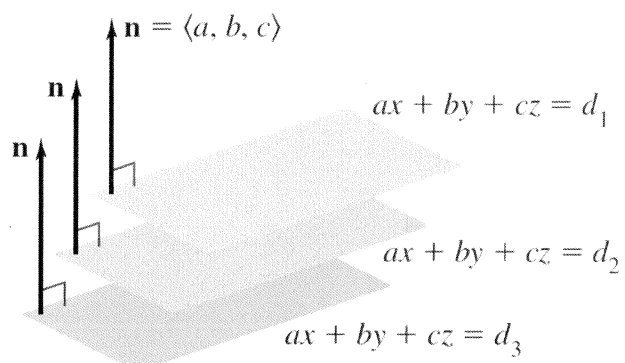
$$\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$$

Elliptic cones

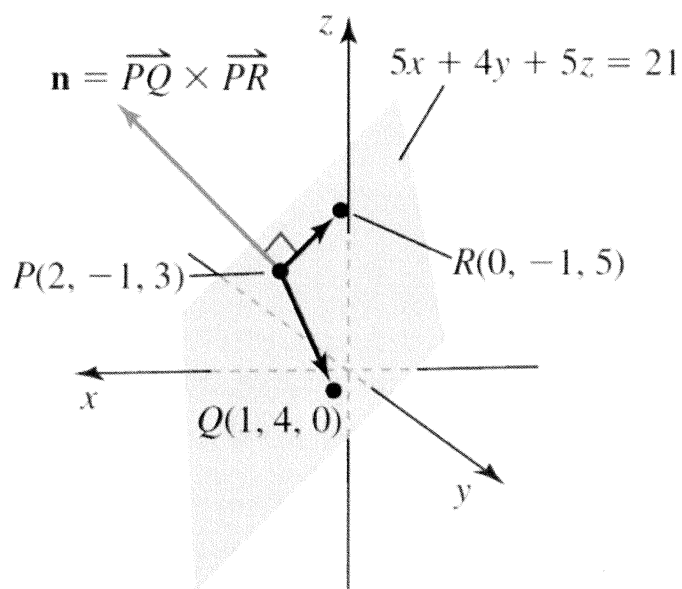
$$\frac{y^2}{4} + z^2 = 4x^2$$



The orientation of a plane is specified by a normal vector $\mathbf{n}$. All vectors $\overrightarrow{P_0P}$ in the plane are orthogonal to $\mathbf{n}$.

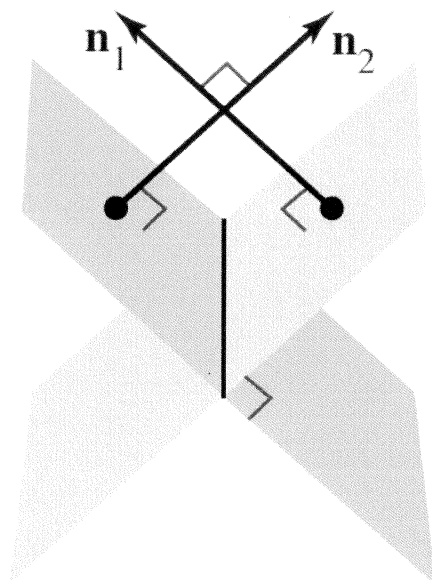
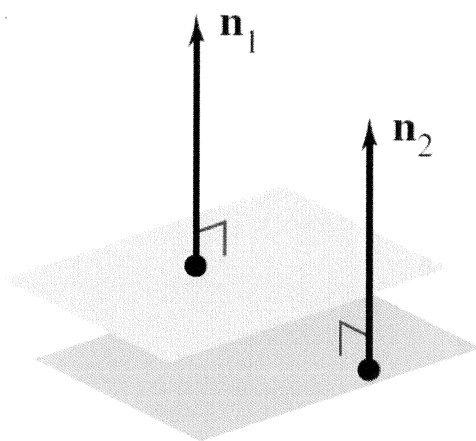


The normal vectors of parallel planes have the same direction.

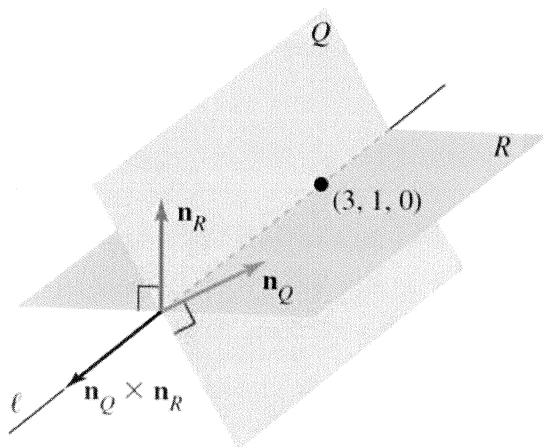


$\overrightarrow{PQ}$ and $\overrightarrow{PR}$ lie in the same plane.
 $\overrightarrow{PQ} \times \overrightarrow{PR}$ is orthogonal to the plane.

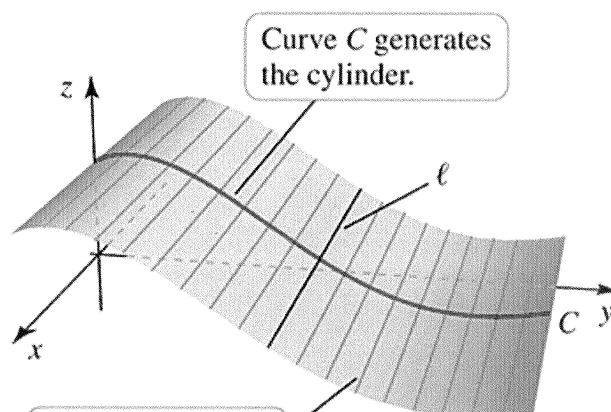
Two distinct planes are parallel if $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel.



Two planes are orthogonal if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.

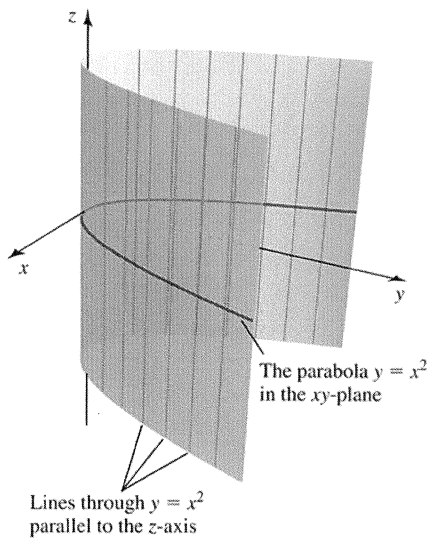


$\mathbf{n}_Q \times \mathbf{n}_R$ is a vector perpendicular to $\mathbf{n}_Q$ and $\mathbf{n}_R$.
Line ℓ is perpendicular to $\mathbf{n}_Q$ and $\mathbf{n}_R$.
Thus, ℓ and $\mathbf{n}_Q \times \mathbf{n}_R$ are parallel to each other.

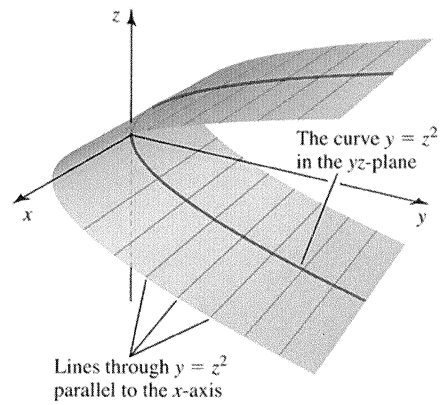


Curve C generates the cylinder.

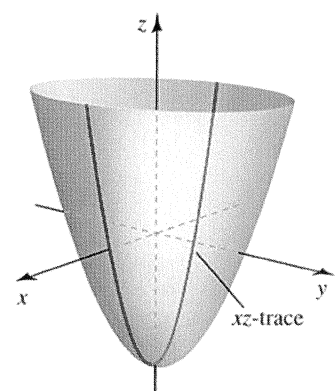
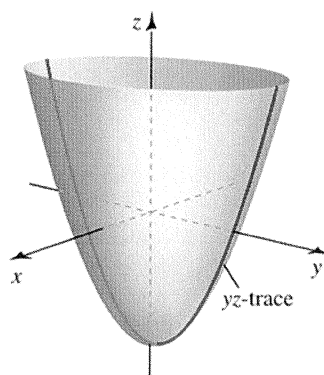
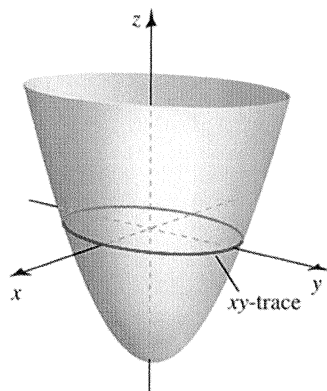
Lines parallel to ℓ through C form the surface.



(a)



(b)



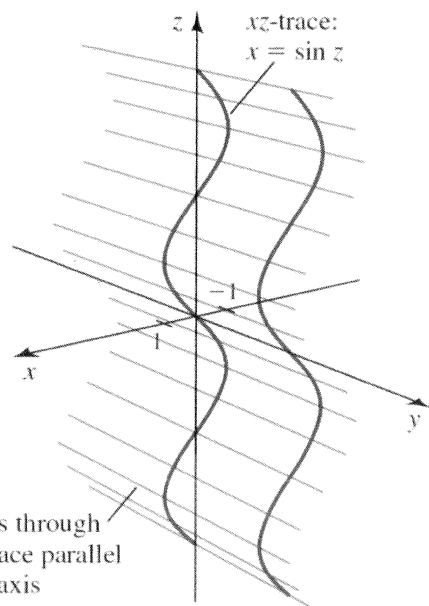
2. Draw a second trace in a plane parallel to the basic trace.

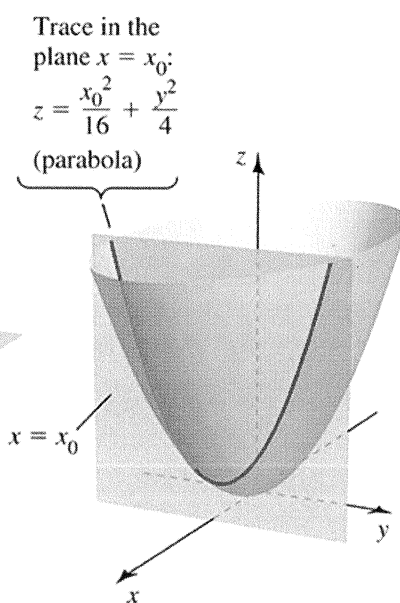
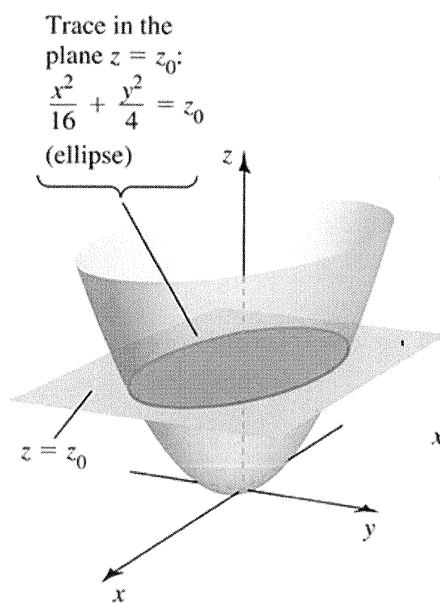
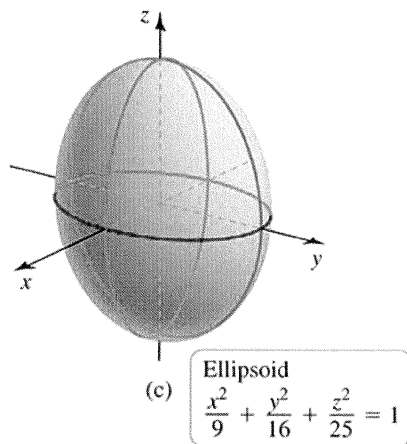
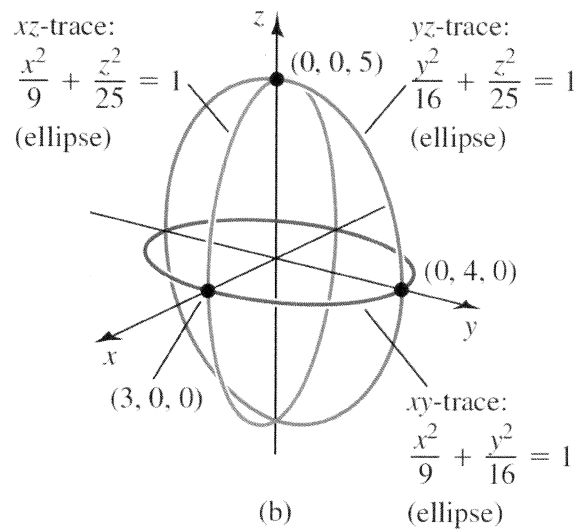
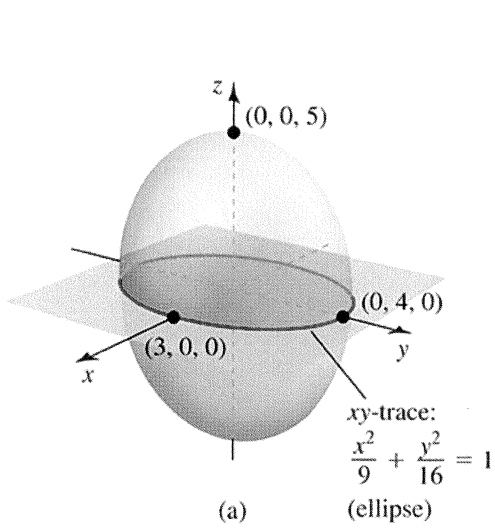
3. Draw parallel lines through the two traces.

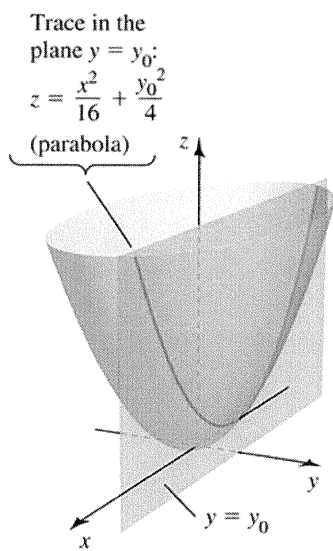
xy -trace:
 $x^2 + 4y^2 = 16$

1. Sketch the basic trace in the appropriate plane.

4. To give definition to the cylinder, draw light outer edges parallel to traces.

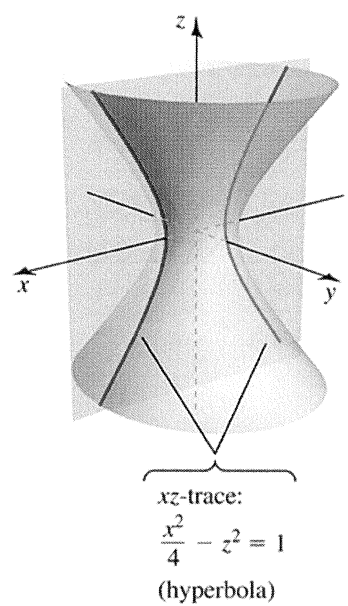
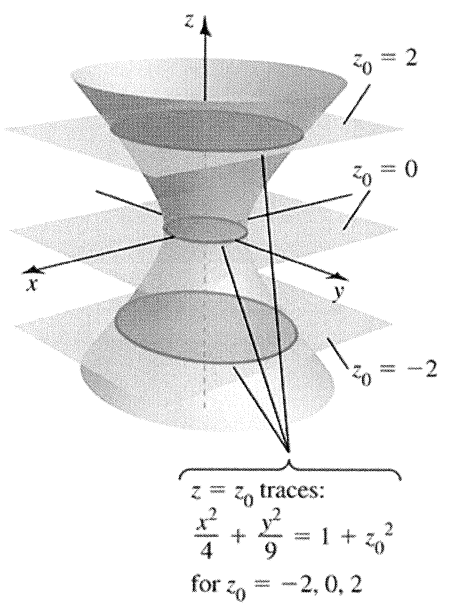
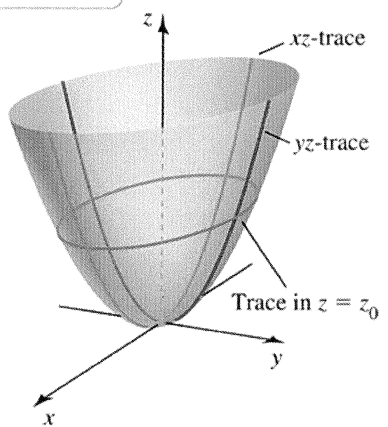


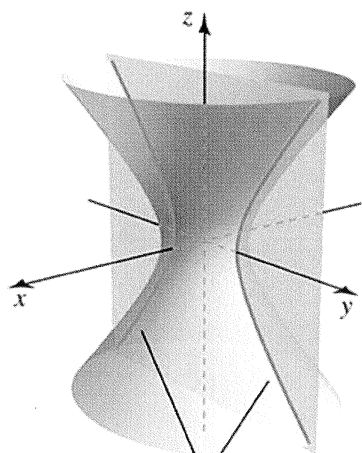




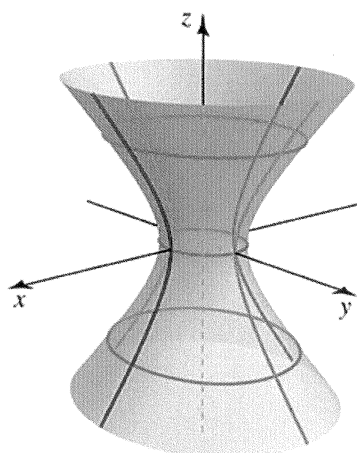
Elliptic paraboloid

$$z = \frac{x^2}{16} + \frac{y^2}{4}$$





yz-trace:
 $\frac{y^2}{9} - z^2 = 1$
 (hyperbola)



Hyperboloid of
 one sheet
 $\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$

