

13.3 Limits and continuity

DEFINITION Limit of a Function of Two Variables

The function f has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$, written

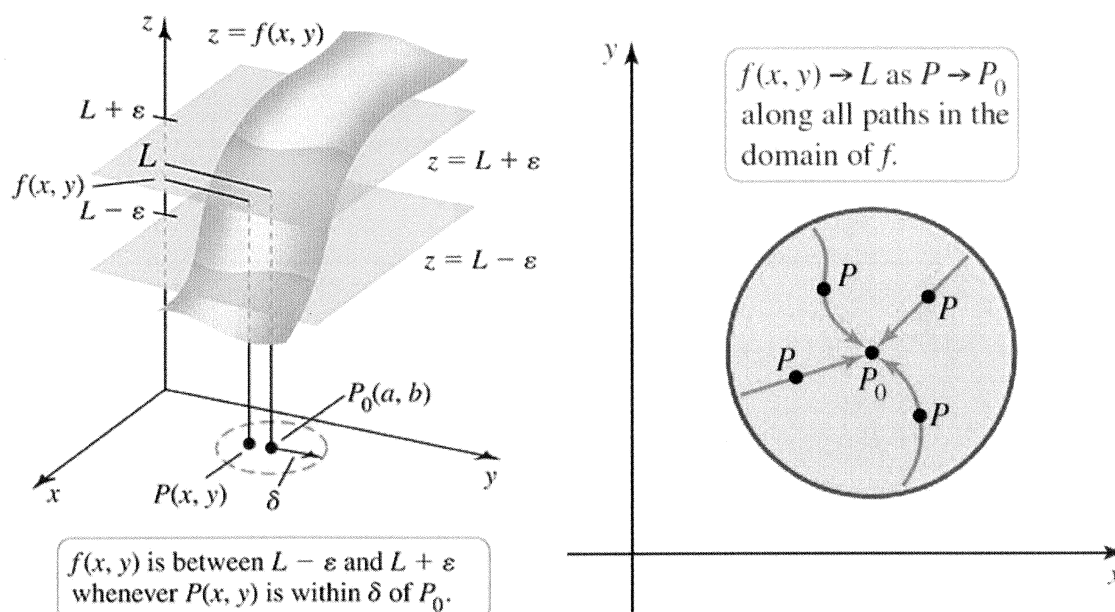
$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = \lim_{P \rightarrow P_0} f(x, y) = L,$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon$$

whenever (x, y) is in the domain of f and

$$0 < |PP_0| = \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$



THEOREM 13.1 Limits of Constant and Linear Functions

Let a , b , and c be real numbers.

1. Constant function $f(x, y) = c$: $\lim_{(x,y) \rightarrow (a,b)} c = c$
2. Linear function $f(x, y) = x$: $\lim_{(x,y) \rightarrow (a,b)} x = a$
3. Linear function $f(x, y) = y$: $\lim_{(x,y) \rightarrow (a,b)} y = b$

THEOREM 13.2 Limit Laws for Functions of Two Variables

Let L and M be real numbers and suppose that $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ and $\lim_{(x,y) \rightarrow (a,b)} g(x,y) = M$. Assume c is a constant, and m and n are integers.

1. Sum $\lim_{(x,y) \rightarrow (a,b)} (f(x,y) + g(x,y)) = L + M$

2. Difference $\lim_{(x,y) \rightarrow (a,b)} (f(x,y) - g(x,y)) = L - M$

3. Constant multiple $\lim_{(x,y) \rightarrow (a,b)} cf(x,y) = cL$

4. Product $\lim_{(x,y) \rightarrow (a,b)} f(x,y)g(x,y) = LM$

5. Quotient $\lim_{(x,y) \rightarrow (a,b)} \left[\frac{f(x,y)}{g(x,y)} \right] = \frac{L}{M}$, provided $M \neq 0$

6. Power $\lim_{(x,y) \rightarrow (a,b)} (f(x,y))^n = L^n$

7. m/n power If m and n have no common factors and $n \neq 0$, then $\lim_{(x,y) \rightarrow (a,b)} [f(x,y)]^{m/n} = L^{m/n}$, where we assume $L > 0$ if n is even.

Ex. $\lim_{(x,y) \rightarrow (2,8)} (3x^2y + \sqrt{xy})$

$$\begin{aligned}
 &= \lim_{(x,y) \rightarrow (2,8)} 3x^2y + \lim_{(x,y) \rightarrow (2,8)} \sqrt{xy} \quad \text{Law 1.} \\
 &= 3 \left[\lim_{(x,y) \rightarrow (2,8)} x \right]^2 \left[\lim_{(x,y) \rightarrow (2,8)} y \right] \quad \text{Laws 3, 4, 6, 7} \\
 &\quad + \sqrt{\left[\lim_{(x,y) \rightarrow (2,8)} x \right] \left[\lim_{(x,y) \rightarrow (2,8)} y \right]} \\
 &= 3(2^2)(8) + \sqrt{(2)(8)} = 100
 \end{aligned}$$

Limits at boundary points

DEFINITION Interior and Boundary Points

Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R , which means it is possible to find a disk centered at P that contains only points of R (Figure 13.40).

A **boundary point** Q of R lies on the edge of R in the sense that every disk centered at Q contains at least one point in R and at least one point not in R .

DEFINITION Open and Closed Sets

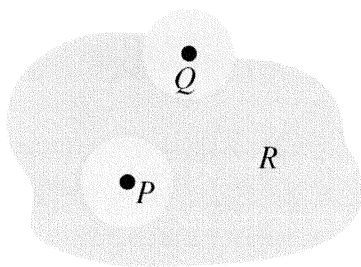
Boundary points : None or All.
"open" "closed"

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

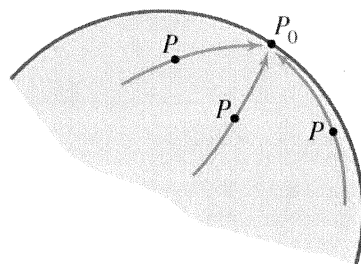
Ex. R defined by $x^2 + y^2 < 9$. Boundary points of R lie on the circle $x^2 + y^2 = 9$. Interior points lie inside the circle and satisfy $x^2 + y^2 < 9$.

$\{(x, y) : x^2 + y^2 < 9\}$: Open , $\{(x, y) : |x| \leq 1, |y| \leq 1\}$: Closed.

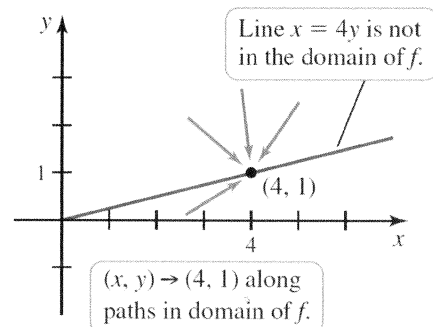
Q is a boundary point:
Every disk centered at Q contains points in R and points not in R .



P is an interior point:
There is a disk centered at P that lies entirely in R .



P must approach P_0 along all paths in the domain of f .



$$\begin{aligned}
 \text{Ex. } \lim_{(x, y) \rightarrow (4, 1)} \frac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}} &= \lim_{(x, y) \rightarrow (4, 1)} \frac{(xy - 4y^2)(\sqrt{x} + 2\sqrt{y})}{(\sqrt{x} - 2\sqrt{y})(\sqrt{x} + 2\sqrt{y})} \\
 &= \lim_{(x, y) \rightarrow (4, 1)} \frac{y(x - 4y)(\sqrt{x} + 2\sqrt{y})}{x - 4y} = \lim_{(x, y) \rightarrow (4, 1)} y(\sqrt{x} + 2\sqrt{y}) \\
 &= 4
 \end{aligned}$$

PROCEDURE Two-Path Test for Nonexistence of Limits

If $f(x, y)$ approaches two different values as (x, y) approaches (a, b) along two different paths in the domain of f , then $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ does not exist

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{(x+y)^2}{x^2+y^2} &= \lim_{x \rightarrow 0} \frac{(x+mx)^2}{x^2+(mx)^2} \\ &= \lim_{x \rightarrow 0} \frac{(1+m)^2 x^2}{(1+m^2)x^2} = \frac{(1+m)^2}{1+m^2} \end{aligned}$$

The limit does NOT exist!

Let (x, y) approach $(0, 0)$ along $y = mx$: m -dependent answer :
No limit.

Continuity of functions of two variables

DEFINITION Continuity

The function f is continuous at the point (a, b) provided

1. f is defined at (a, b) .
2. $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ exists.
3. $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$

Checking the continuity : $f(x, y) = \begin{cases} \frac{3xy^2}{x^2+y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

$y = mx$:

$$f(x, y) = \frac{3x \cdot (mx)^2}{x^2 + (mx)^4} \rightarrow 0 \quad \text{when } x \rightarrow 0$$

Now, $x = my^2$

$$f(x, y) = \frac{3my^2 y^2}{m^2 y^4 + y^4} = \frac{3m}{m^2 + 1} : \text{depending on } m.$$

No limit.
No continuity
at $(0, 0)$.

Composite functions**THEOREM 13.3 Continuity of Composite Functions**

If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is continuous at (a, b) .

a) $h(x, y) = \ln(x^2 + y^2 + 4)$
 $= f(g(x, y))$
 $f(u) = \ln u$, $u = g(x, y) = x^2 + y^2 + 4$
 $f(u)$ is continuous for $u > 0$ and $u = x^2 + y^2 + 4 > 0$
 for all (x, y) . So, $h(x, y)$ is everywhere continuous

b) $h(x, y) = e^{x/y} = f(g(x, y))$
 $f(u) = e^u$, $u = g(x, y) = \frac{x}{y}$
 f is continuous for any u in $\mathbb{R}$. g is continuous for all (x, y) provided $y \neq 0$.
 $h(x, y)$ is continuous on $\{(x, y) : y \neq 0\}$.

Functions of three variables

$$\lim_{(x, y, z) \rightarrow (2, \frac{\pi}{2}, 0)} \frac{x^2 \sin y}{z^2 + 4} = \frac{2^2 \sin \frac{\pi}{2}}{0^2 + 4} = 1$$

because of continuity at $(2, \frac{\pi}{2}, 0)$

$$h(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 1}$$

$$= \sqrt{u}, \quad u = x^2 + y^2 + z^2 - 1 = g(x, y, z)$$

$\sqrt{u}$ is continuous for $u \geq 0$.

$g(x, y, z)$ is continuous and non-negative for $x^2 + y^2 + z^2 \geq 1$. So, $h(x, y, z)$ is continuous at all points on or outside the unit sphere.