

13.4 Partial derivatives

Derivatives with two variables

Consider $f(x, y)$ defined on a domain D in the xy plane

△ Partial derivative of f w.r.t. x , evaluated at (a, b)

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

in which y is fixed at $y = b$. $\rightarrow$ fig

△ Partial derivative of f w.r.t. y , evaluated at (a, b)

$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$

in which x is fixed at $x = a$. $\rightarrow$ fig

If we replace (a, b) by the variable point (x, y) , then f_x and f_y become functions of x and y .

$$\frac{\partial f}{\partial x}(a, b) = \left. \frac{\partial f}{\partial x} \right|_{(a, b)} = f_x(a, b)$$

$$\frac{\partial f}{\partial y}(a, b) = \left. \frac{\partial f}{\partial y} \right|_{(a, b)} = f_y(a, b)$$

△ All the rules and results for ordinary derivatives can be used to compute partial derivatives

$f_x(x, y)$: y treated as a constant and take $\frac{d}{dx}$.

$f_y(x, y)$: x treated as a constant and take $\frac{d}{dy}$.

Ex 1 $f(x, y) = x^2 - y^2 + 4$

a) $f_x(x, y) = 2x$,

$f_y(x, y) = -2y$.

b) $f_x(2, -4) = 2(2) = 4$

$f_y(2, -4) = -2(-4) = 8$

Ex 2. a) $f(x, y) = \sin(xy)$

$f_x = y \cos(xy)$, $f_y = x \cos(xy)$

b) $f(x, y) = x^2 e^{xy}$

$f_x = 2x e^{xy} + x^2 y e^{xy} = (2x + x^2 y) e^{xy}$

$f_y = x^2 x e^{xy} = x^3 e^{xy}$

► Higher-order partial derivatives

f_x and f_y are functions of x and y .

$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}$, $(f_x)_x = f_{xx}$

$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}$, $(f_y)_y = f_{yy}$

$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}$, $(f_y)_x = f_{yx}$

$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$, $(f_x)_y = f_{xy}$

Ex. $f(x, y) = 3x^4 y - 2xy + 5xy^3$

$f_x = 12x^3 y - 2y + 5y^3$

$f_y = 3x^4 - 2x + 15xy^2$

$f_{xx} = 36x^2 y$

$f_{yx} = 12x^3 - 2 + 15y^2$

$f_{xy} = 12x^3 - 2 + 15y^2$

$f_{yy} = 30xy$

with $f_{xy} = f_{yx}$

Is this a general result?

Equality of mixed partial derivatives

Assume f is defined on an open set D on $\mathbb{R}^2$, and f_{xy} and f_{yx} are continuous throughout D . Then $f_{xy} = f_{yx}$ at all points of D .

Functions of three variables

$$f(x, y, z) = e^{-xy} \cos z$$

$$\begin{aligned} f_x &= -y e^{-xy} \cos z, & y \text{ and } z \text{ fixed} \\ f_y &= -x e^{-xy} \cos z, & x \text{ and } z \text{ fixed} \\ f_z &= -e^{-xy} \sin z. & x \text{ and } y \text{ fixed} \end{aligned}$$

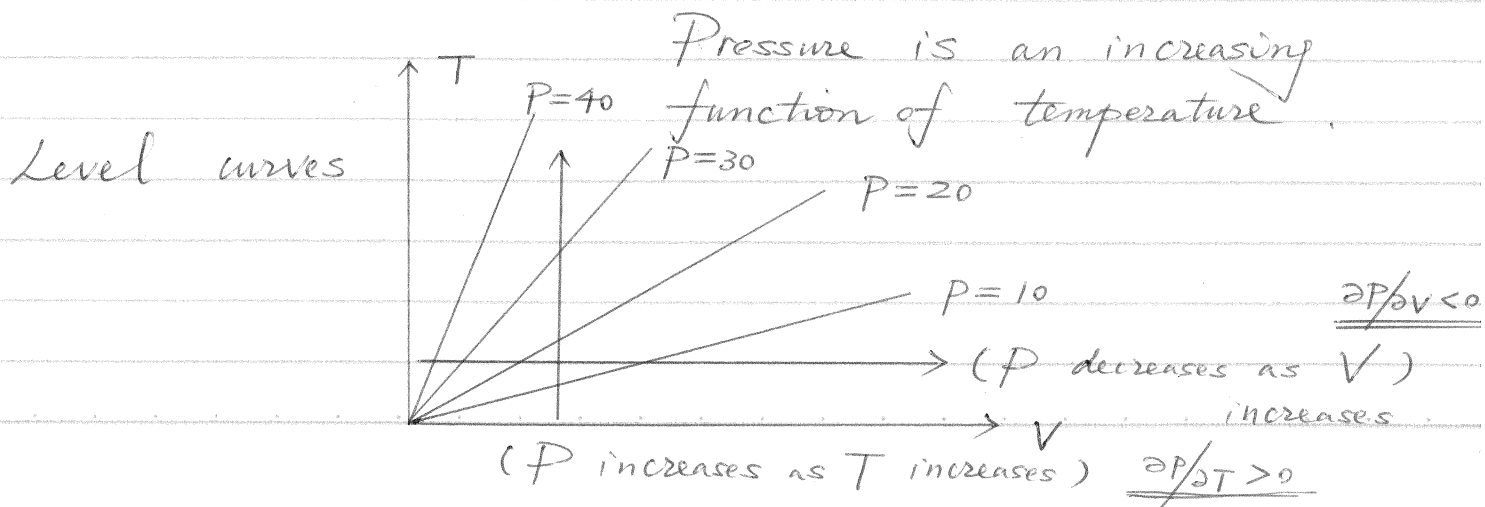
Ex. Ideal Gas Law

$$PV = kT, \quad k > 0 \text{ is a constant.}$$

P : pressure, V : volume, T : temperature

$$P = P(V, T) = k \frac{T}{V}, \quad \text{pressure } P \text{ as a function of } V \text{ and } T.$$

$$\begin{aligned} \frac{\partial P}{\partial V} &= -k \frac{T}{V^2}, & \frac{\partial P}{\partial V} < 0: & \text{Pressure is a decreasing function of Volume.} \\ \frac{\partial P}{\partial T} &= k \frac{1}{V}, & \frac{\partial P}{\partial T} > 0: & \end{aligned}$$



Differentiability

$f(x)$ is differentiable at $x=a$ if the limit

$$f'(a) = \lim_{\Delta x \rightarrow 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$
 exists.

Define $\varepsilon = \frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)$.

If $f(x)$ is differentiable at $x=a$, then

$$\lim_{\Delta x \rightarrow 0} \varepsilon = 0$$

$$\Delta y = f(a+\Delta x) - f(a) = f'(a) \Delta x + \varepsilon \Delta x.$$

The change in $y=f(x)$ is represented by $f'(a) \Delta x$ plus $\varepsilon \Delta x$, with $\lim_{\Delta x \rightarrow 0} \varepsilon = 0$.

• Linear approximation from differentiability.

△ $Z=f(x,y)$ is differentiable at (a,b) provided $f_x(a,b)$ and $f_y(a,b)$ exist and the change

$$\Delta Z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

 equals

$$\Delta Z = f_x(a,b) \Delta x + f_y(a,b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 are functions depending only on Δx and Δy , with $(\varepsilon_1, \varepsilon_2) \rightarrow (0,0)$ as $(\Delta x, \Delta y) \rightarrow (0,0)$.

△ A function is differentiable on an OPEN set R if it is differentiable at every point of R .

△ Suppose f has f_x and f_y defined on an open set containing (a,b) , with f_x and f_y CONTINUOUS at (a,b) . Then f is differentiable at (a,b) .

Note: Existence of f_x and f_y at (a,b) is NOT enough.

Differentiability implies continuity.

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \Delta Z = 0$$

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} f(a+\Delta x, b+\Delta y) = f(a,b) \quad \downarrow$$

continuity
at (a,b)

Ex. $f(x,y) = \begin{cases} \frac{3xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$y = mx$. $\lim_{(x,y) \rightarrow (0,0)} \frac{3xy}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{3mx^2}{x^2+m^2x^2}$

$$= \frac{3m}{1+m^2}$$

m -dependence $\rightarrow$ No limit $\rightarrow$

Not continuous

$\rightarrow$ Not differentiable

No differentiability at $(0,0)$.

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0-0}{h} = 0$$

However, f_x and f_y exist at $(0,0)$.

But f_x and f_y are not continuous at $(0,0)$.

