

13.5 The chain rule

$$y = y(u) \quad \text{and} \quad u = u(t) \quad : \quad \frac{dy}{dt} = \frac{dy}{du} \frac{du}{dt}$$

The chain rule with one independent variable
 $z = f(x, y)$ where x and y are functions of t .

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$\Delta z = f_x(x, y) \Delta x + f_y(x, y) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{\Delta z}{\Delta t} &= f_x(x, y) \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} + f_y(x, y) \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} \\ &\quad + \lim_{\Delta t \rightarrow 0} \varepsilon_1 \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \rightarrow 0} \varepsilon_2 \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} \end{aligned}$$

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \quad \text{with } \varepsilon_1 \text{ and } \varepsilon_2 \rightarrow 0$$

for $\Delta t \rightarrow 0$, due to the differentiability of $f(x, y)$.

$$\begin{aligned} \frac{dz}{dt} &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \\ &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \end{aligned}$$

$w = f(x, y, z)$ where x, y , and z are functions of the single independent variable t .

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

Ex.

$$z = x^2 - 3y^2 + 20 \quad \text{where } x = 2 \cos t \text{ and } y = 2 \sin t.$$

$$\frac{dz}{dt} = (2x)(-2 \sin t) + (-6y)(2 \cos t)$$

$$= -8 \sin t \cos t - 24 \sin t \cos t = -16 \sin 2t.$$

$$\left. \frac{dz}{dt} \right|_{t=\pi/4} = -16 \sin \frac{\pi}{2} = -16.$$

For a geometrical picture, see figure.

The chain rule with several independent variables.
 z is a function of x and y , where x and y are functions of s and t .
 (All functions are differentiable.)

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

Ex. $z = \sin 2x \cos 3y$

$$x = s+t, \quad y = s-t$$

$$\frac{\partial z}{\partial s} = (2 \cos 2x \cos 3y)(1) + (-3 \sin 2x \sin 3y)(1)$$

$$= 2 \cos 2x \cos 3y - 3 \sin 2x \sin 3y$$

$$= 2 \cos(2(s+t)) \cos(3(s-t)) - 3 \sin(2(s+t)) \sin(3(s-t)).$$

$$\frac{\partial z}{\partial t} = (2 \cos 2x \cos 3y)(1) + (-3 \sin 2x \sin 3y)(-1)$$

$$= 2 \cos 2x \cos 3y + 3 \sin 2x \sin 3y$$

Implicit differentiation

x and y are related by $F(x, y) = 0$.
 y is implicitly defined as a function of x .

$$\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0 \quad \text{from } \frac{d}{dx} F(x, y(x)) = 0.$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \quad \text{provided } F_y \neq 0.$$

Ex. $F(x, y) = \sin xy + \pi y^2 - x = 0$

$$F_x = y \cos xy - 1$$

$$F_y = x \cos xy + 2\pi y$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y \cos xy - 1}{x \cos xy + 2\pi y}$$

Ex. Fluid Flow.

$$u = u(x, y) = 2 \sin \pi x \cos \pi y$$

$$v = v(x, y) = -2 \cos \pi x \sin \pi y$$

$$\text{Speed } S = S(x, y) = \sqrt{u(x, y)^2 + v(x, y)^2}$$

Find $\partial S / \partial x$ and $\partial S / \partial y$

$$\frac{\partial S}{\partial u} = \frac{2u}{2\sqrt{u^2 + v^2}} = \frac{u}{S}, \quad \frac{\partial S}{\partial v} = \frac{v}{S}$$

$$\frac{\partial S}{\partial x} = \frac{\partial S}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial S}{\partial v} \frac{\partial v}{\partial x}$$

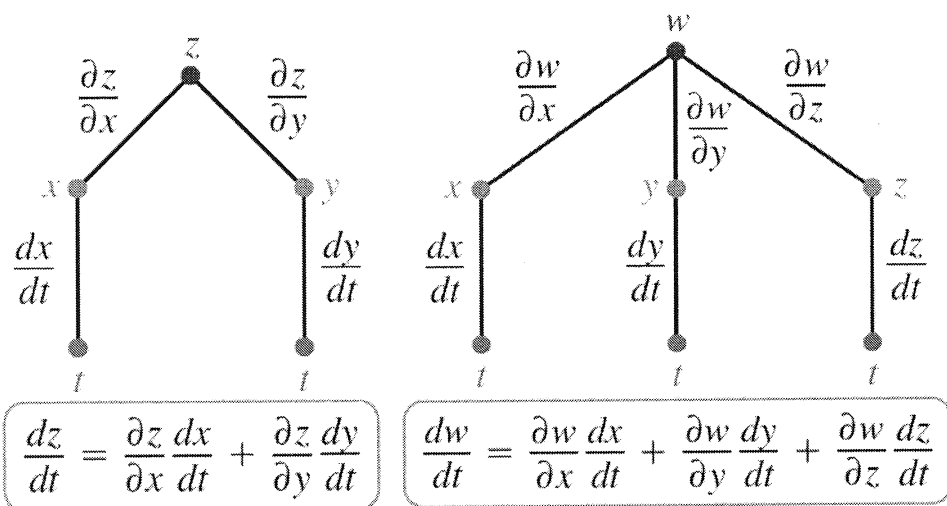
$$= \frac{u}{S} (2\pi \cos \pi x \cos \pi y) + \frac{v}{S} (-2\pi \sin \pi x \sin \pi y)$$

$$= \frac{2\pi}{S} (u \cos \pi x \cos \pi y - v \sin \pi x \sin \pi y)$$

Similarly

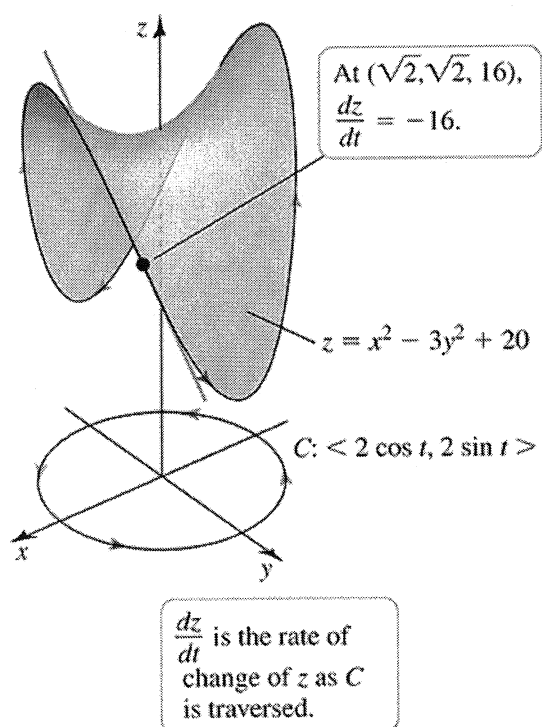
$$\frac{\partial S}{\partial y} = \frac{u}{S} (-2\pi \sin \pi x \sin \pi y) + \frac{v}{S} (-2\pi \cos \pi x \cos \pi y)$$

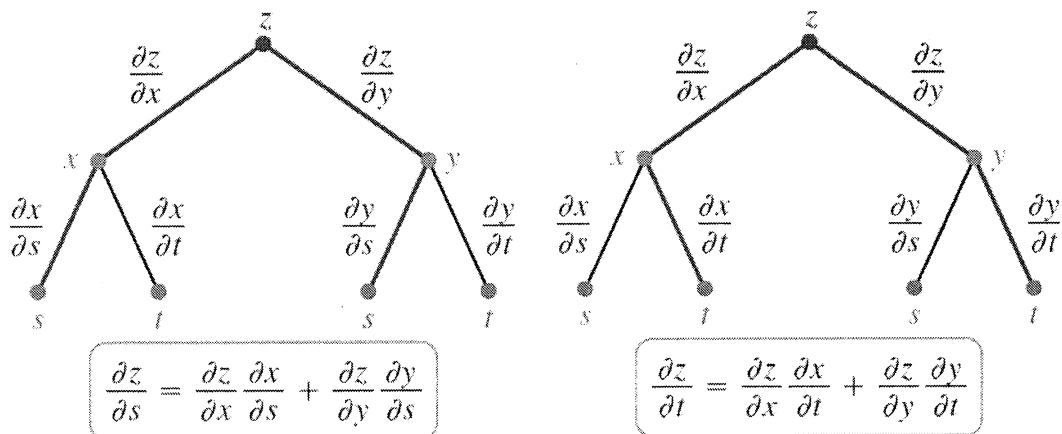
$$= -\frac{2\pi}{S} (u \sin \pi x \sin \pi y + v \cos \pi x \cos \pi y)$$

**THEOREM 13.7 Chain Rule (One Independent Variable)**

Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I . Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$





THEOREM 13.8 Chain Rule (Two Independent Variables)

Let z be a differentiable function of x and y , where x and y are differentiable functions of s and t . Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}.$$

THEOREM 13.9 Implicit Differentiation

Let F be differentiable on its domain and suppose that $F(x, y) = 0$ defines y as a differentiable function of x . Provided $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

