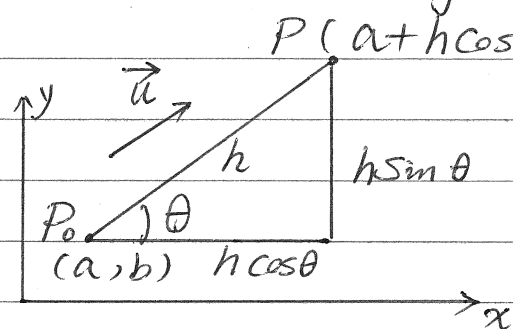


13.6 Directional derivatives and the gradient

Directional derivatives

Let $(a, b, f(a, b))$ be a point on the surface of $z = f(x, y)$ and let $\vec{u} = \langle u_1, u_2 \rangle = \langle \cos \theta, \sin \theta \rangle$ be a unit vector in the xy plane

- The rate of change of $f(x, y)$ at $P_0(a, b)$ in the direction of $\vec{u}$



(x, y) : from P_0 to P
Slope of the secant line connecting two points on the surface corresponding to P_0 and P .
 $\Rightarrow$ figure

$(a, b, f(a, b))$ and $(a + h \cos \theta, b + h \sin \theta, f(a + h \cos \theta, b + h \sin \theta))$

$$\text{slope} = \frac{\Delta z}{h} = \frac{f(a + h \cos \theta, b + h \sin \theta) - f(a, b)}{h}$$

Δz : change in z , h : distance in the xy plane.

Definition: Directional derivative of f at (a, b) in the direction of $\vec{u} = \langle \cos \theta, \sin \theta \rangle$

$$D_{\vec{u}} f(a, b) = \lim_{h \rightarrow 0} \frac{f(a + h \cos \theta, b + h \sin \theta) - f(a, b)}{h}$$

$$g(s) = f(a + s \cos \theta, b + s \sin \theta) = f(x(s), y(s))$$

$$D_{\vec{u}} f(a, b) = \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = g'(0)$$

$$\begin{aligned} \text{Chain rule: } g'(0) &= f_x(a, b) \left. \frac{dx}{ds} \right|_{s=0} + f_y(a, b) \left. \frac{dy}{ds} \right|_{s=0} \\ &= f_x(a, b) \cos \theta + f_y(a, b) \sin \theta \end{aligned}$$

$$= \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle$$

Ex 1. $z = f(x, y) = \frac{1}{4}x^2 + \frac{1}{2}y^2 + 2$. paraboloid.

$$\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle, \quad \vec{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$$

Find $D_{\vec{u}}f$ and $D_{\vec{v}}f$ at $P_0(3, 2)$

$$f_x = \frac{x}{2}, \quad f_y = y$$

$$\langle f_x(P_0), f_y(P_0) \rangle = \left\langle \frac{3}{2}, 2 \right\rangle$$

$\vec{u}$ and $\vec{v}$ are already unit vectors

$$D_{\vec{u}}f(3, 2) = \left\langle \frac{3}{2}, 2 \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{7}{2\sqrt{2}}$$

$$\begin{aligned} D_{\vec{v}}f(3, 2) &= \left\langle \frac{3}{2}, 2 \right\rangle \cdot \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle \\ &= \frac{3}{4} - \sqrt{3} \end{aligned}$$

Δ The gradient vector

Let f be differentiable at (x, y) . The gradient of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = f_x(x, y)\vec{i} + f_y(x, y)\vec{j}$$

$$D_{\vec{u}}f(x, y) = \nabla f(x, y) \cdot \vec{u} \quad \vec{u}: \text{unit vector.}$$

Ex 2. $f(x, y) = x^2 + 2xy - y^3$

$$\nabla f(x, y) = f_x\vec{i} + f_y\vec{j} = (2x + 2y)\vec{i} + (2x - 3y^2)\vec{j}$$

$$\nabla f(3, 2) = 10\vec{i} - 6\vec{j}$$

Ex 3. $f(x, y) = 3 - \frac{x^2}{10} + \frac{xy^2}{10}$

Compute the directional derivative of f at $(3, -1)$ in the direction of $\vec{u} = \langle 3, 4 \rangle$.

$$f_x = -\frac{x}{5} + \frac{y^2}{10}, \quad f_y = \frac{xy}{5}$$

$$f_x(3, -1) = -\frac{3}{5} + \frac{1}{10} = -\frac{5}{10} = -\frac{1}{2}, \quad f_y(3, -1) = \frac{-3}{5}$$

$$\vec{v} = \vec{u}/|\vec{u}| = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$$\begin{aligned} D_{\vec{v}}f(3, -1) &= \nabla f(3, -1) \cdot \vec{v} = \left\langle -\frac{1}{2}, -\frac{3}{5} \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\ &= -\frac{39}{50} \end{aligned}$$

Δ Interpretations of the gradient

Interpretation based on $D_{\vec{u}} f(a, b) = \nabla f(a, b) \cdot \vec{u}$
 $= |\nabla f(a, b)| \cos \alpha$

α is the angle between $\nabla f(a, b)$ and $\vec{u}$.
 $|\vec{u}| = 1$

1. Maximum rate of increase if $\alpha = 0$

$D_{\vec{u}} f(a, b) = |\nabla f(a, b)| > 0$ if $\vec{u}$ is in the direction of $\nabla f(a, b)$.

2. Maximum rate of decrease if $\alpha = \pi$

$D_{\vec{u}} f(a, b) = -|\nabla f(a, b)| < 0$ if $\vec{u}$ is in the direction of $-\nabla f(a, b)$

Ex. Steepest ascent and descent

$z = f(x, y) = 4 + x^2 + 3y^2$

$(2, -\frac{1}{2}, \frac{35}{4})$ is on the paraboloid.

$\nabla f(2, -\frac{1}{2}) = \langle f_x(2, -\frac{1}{2}), f_y(2, -\frac{1}{2}) \rangle$
 $= \langle 4, -3 \rangle$

Steepest ascent in the direction of $\langle 4, -3 \rangle$
 or $\langle \frac{4}{5}, -\frac{3}{5} \rangle$

Steepest descent in the direction of $-\langle 4, -3 \rangle$
 or $\langle -\frac{4}{5}, \frac{3}{5} \rangle$

Δ The gradient and level curves

A level curve of $z = f(x, y)$ is a curve in the xy plane defined by $f(x, y) = z_0$.

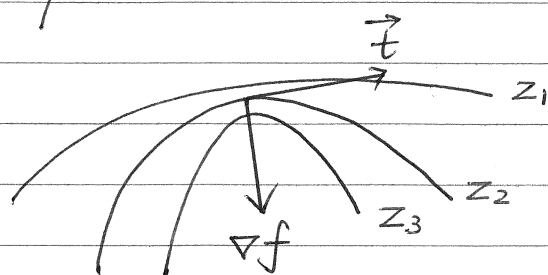
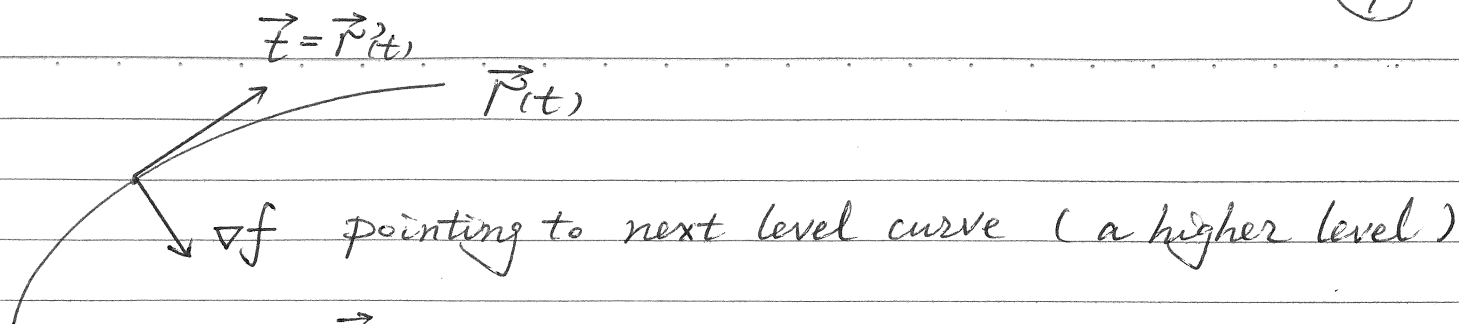
1. Suppose the curve is parametrized by t :
 $x = x(t)$, $y = y(t)$ with $f(x(t), y(t)) = z_0$.

2. Chain rule: $\frac{d}{dt} f(x(t), y(t)) = 0 \rightarrow$

$f_x(x, y) x'(t) + f_y(x, y) y'(t) = 0$

$\langle f_x, f_y \rangle \cdot \langle x', y' \rangle = 0$, i.e., $\nabla f \cdot \frac{d\vec{r}}{dt} = 0$

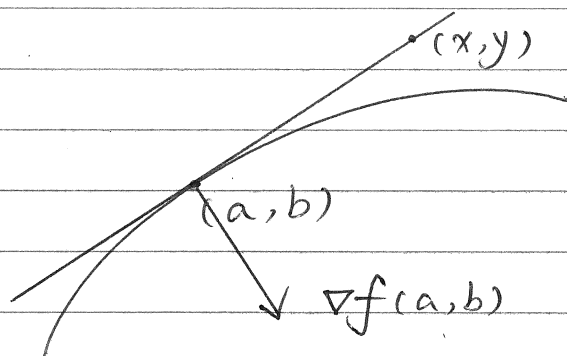
The tangent vector $\vec{t} = \frac{d\vec{r}}{dt}$ is orthogonal to ∇f .



levels $z_1 < z_2 < z_3$

∇f in the direction of steepest ascent.

∇f is locally orthogonal to the level curve.



$$\nabla f(a, b) \cdot \langle x-a, y-b \rangle = 0$$

gradient $\cdot$ tangent vector = 0

$$f_x(a, b)(x-a) + f_y(a, b)(y-b) = 0$$

Equation of the tangent line.

Ex. $z = f(x, y) = \sqrt{1 + 2x^2 + y^2}$

Upper sheet of a hyperboloid of two sheets.

a) $\nabla f(1, 1)$

$$f_x = \frac{2x}{\sqrt{1 + 2x^2 + y^2}}$$

$$f_y = \frac{y}{\sqrt{1 + 2x^2 + y^2}}$$

$$\nabla f(1, 1) = \vec{i} + \frac{1}{2}\vec{j}$$

Level curve through $(1, 1, 2)$

$$z = \sqrt{1 + 2x^2 + y^2}$$

$$2x^2 + y^2 = 3$$

Ellipse

d/dx : $4x + 2yy' = 0$, $y' = -\frac{2x}{y} = -2$

Slope of the tangent line is -2.

$\vec{r} = \langle 1, -2 \rangle$ or any of its scalar multiple.

$$\nabla f(1, 1) \cdot \vec{r} = \langle 1, \frac{1}{2} \rangle \cdot \langle 1, -2 \rangle = 0$$

Orthogonality between ∇f and $\vec{r}$ verified.

b) Equation of the tangent line at $(1, 1)$

$$f_x(1, 1)(x-1) + f_y(1, 1)(y-1) = 0$$

$$y = -2x + 3 \quad \text{Slope} = -2.$$

△ Gradient in three dimensions

$$W = f(x, y, z)$$

Gradient of f at (x, y, z) :

$$\nabla f(x, y, z) = f_x(x, y, z)\vec{i} + f_y(x, y, z)\vec{j} + f_z(x, y, z)\vec{k}$$

Directional derivative of f at (a, b, c) in the direction of unit vector $\vec{u} = \langle u_1, u_2, u_3 \rangle$:

$$D_{\vec{u}} f(a, b, c) = \nabla f(a, b, c) \cdot \vec{u}$$

Ex. $W = f(x, y, z) = x^2 + 2y^2 + 4z^2 - 1$

Level surface $f(x, y, z) = 3$

$$x^2 + 2y^2 + 4z^2 = 4$$

An ellipsoid

Intercepts : $(2, 0, 0)$, $(0, \sqrt{2}, 0)$, $(0, 0, 1)$

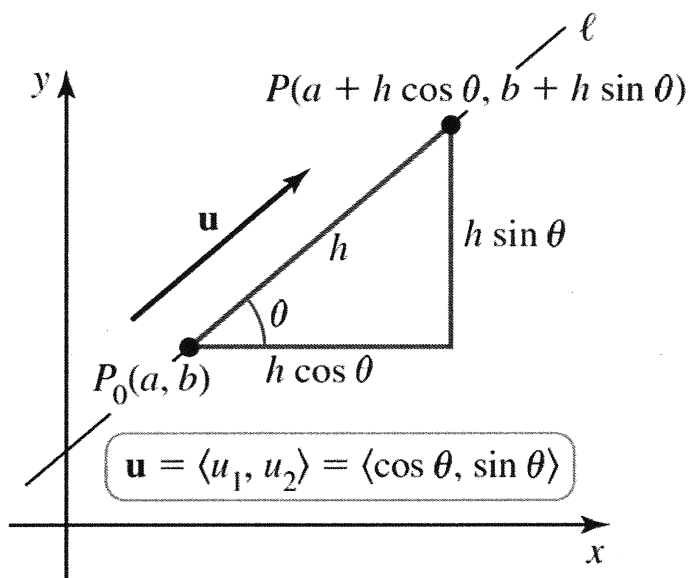
$$\nabla f = \langle 2x, 4y, 8z \rangle$$

$$\nabla f(2, 0, 0) = 4\vec{i}$$

$$\nabla f(0, \sqrt{2}, 0) = 4\sqrt{2}\vec{j}$$

$$\nabla f(0, 0, 1) = 8\vec{k}$$

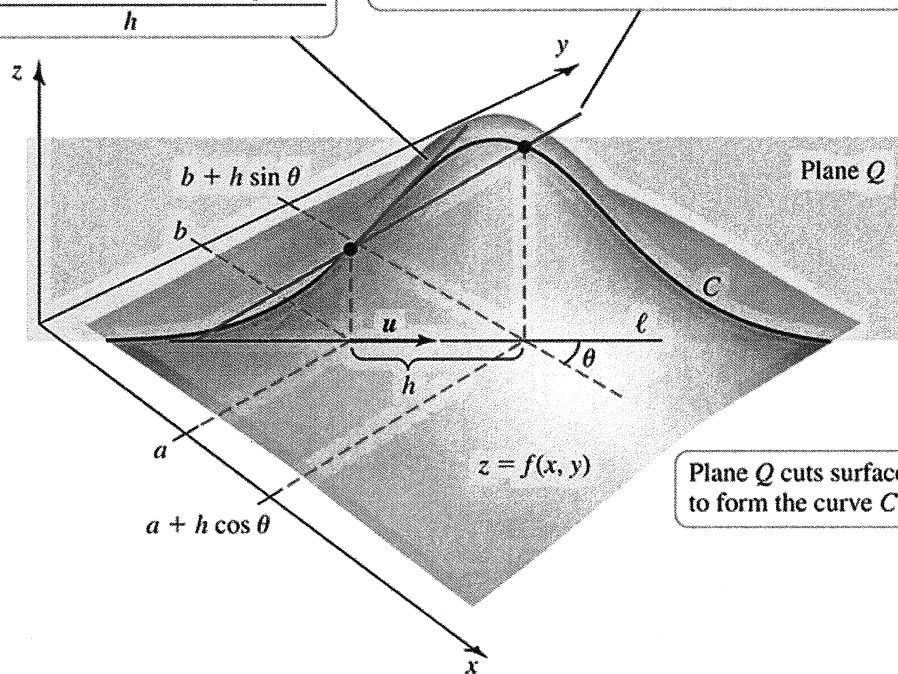
At each point, ∇f is orthogonal to the level surface, to be made clear in next sec.

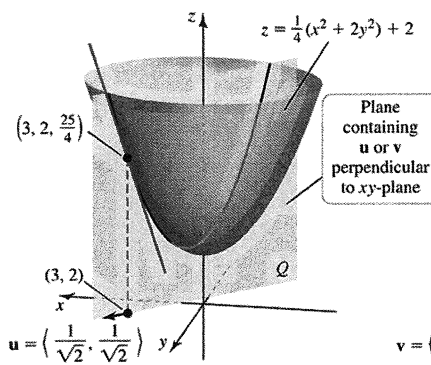


Slope of tangent line

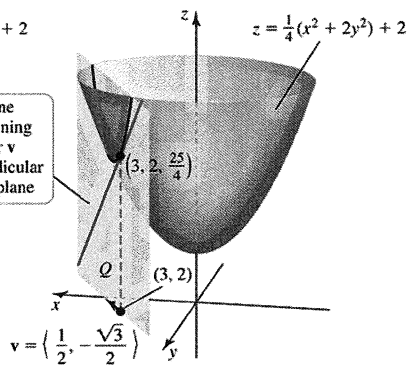
$$= \lim_{h \rightarrow 0} \frac{f(a + h \cos \theta, b + h \sin \theta) - f(a, b)}{h}$$

Slope of secant line = $\frac{f(a + h \cos \theta, b + h \sin \theta) - f(a, b)}{h}$

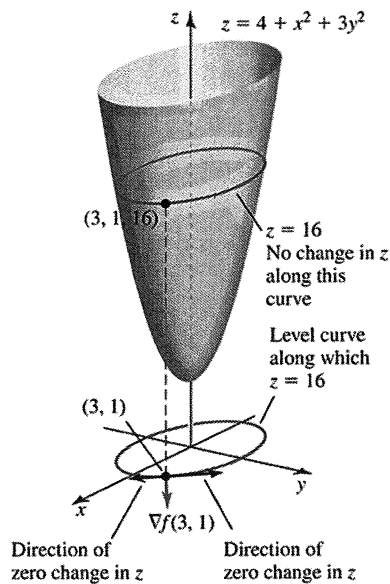
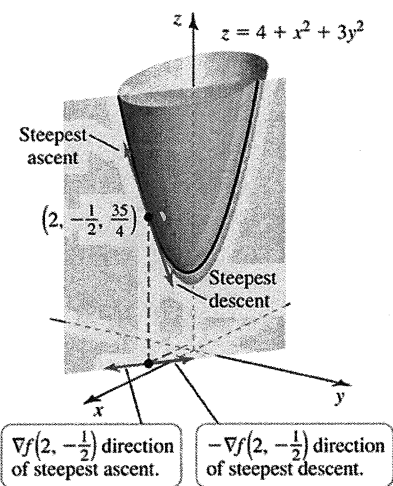
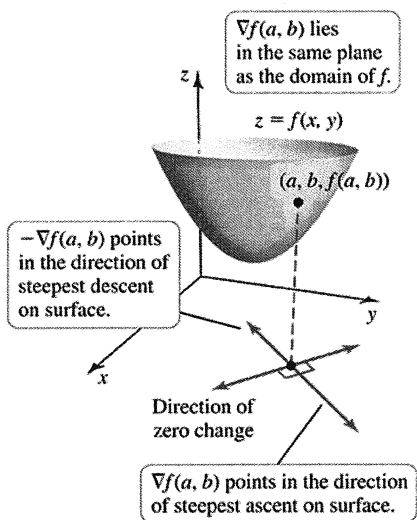


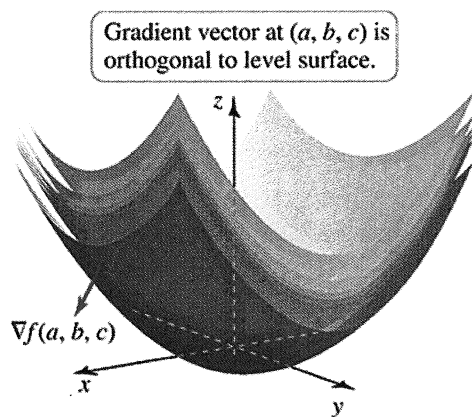
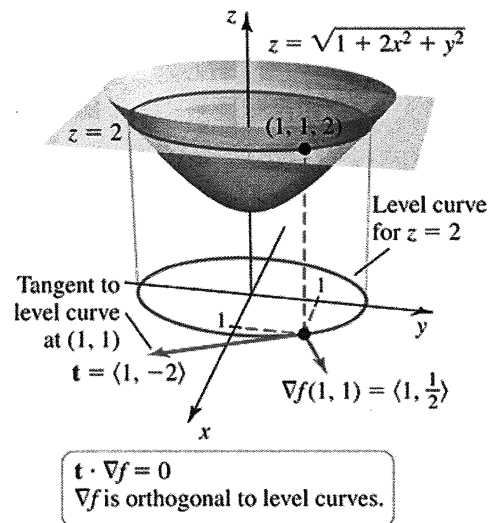
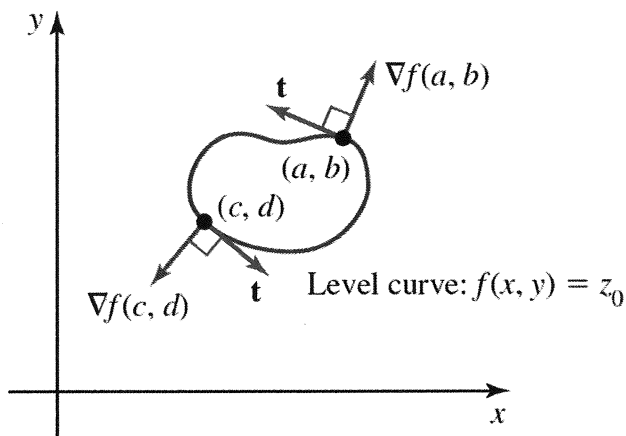


Slope of line in Q tangent to the intersection curve in the direction of u has slope $D_u f \approx 2.47$.



Slope of line in Q tangent to the intersection curve in the direction of v has slope $D_v f \approx -0.98$.





Level surface of $f(x, y, z) = x^2 + 2y^2 + 4z^2 - 1$
 $f(x, y, z) = 3$

