

13.7

Tangent planes and linear approximation

 $z = f(x, y)$ is differentiable $\rightarrow$ A smooth surfaceFocus on a single point: In the vicinity of that point, the surface appears like a plane Δ Tangent planes for $F(x, y, z) = 0$
A surface implicitly defined $\vec{r} = \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a smooth curve on the surface. $\rightarrow F(x(t), y(t), z(t)) = 0$

$$\begin{aligned} \frac{d}{dt} F(x(t), y(t), z(t)) &= F_x x' + F_y y' + F_z z' \\ &= \langle F_x, F_y, F_z \rangle \cdot \langle x', y', z' \rangle \\ &= \nabla F(x, y, z) \cdot \vec{r}'(t) \end{aligned}$$

Orthogonality between the tangent vector $\vec{r}'(t)$ and the gradient.Now, fix a point (a, b, c) Any smooth curve has $\nabla F \cdot \vec{r}' = 0$ at (a, b, c) .
Tangent vectors of all these curves are orthogonal to ∇F : These tangent vectors all lie in the same plane.
This is the TANGENT PLANE.Normal vector $\nabla F(a, b, c)$:

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0$$

as the equation of the tangent plane.

See figure 13.78

The tangent plane contains all the tangent vectors defined at a given point (a, b, c) .

Ex $F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0$ is an ellipsoid.
Tangent plane at $(0, 4, \frac{3}{5})$.

$$\nabla F(0, 4, \frac{3}{5}) = \left\langle \frac{2x}{9}, \frac{2y}{25}, 2z \right\rangle \bigg|_{(0, 4, \frac{3}{5})} = \left\langle 0, \frac{8}{25}, \frac{6}{5} \right\rangle$$

$$0(x-0) + \frac{8}{25}(y-4) + \frac{6}{5}(z-\frac{3}{5}) = 0$$

$$4y + 15z = 25.$$

At what points the T.P. is horizontal.

Horizontal T.P. : ∇F is parallel to $\vec{k}$.

$x=y=0$ in $\nabla F = \langle 0, 0, 2z \rangle$ with $z = \pm 1$.

$(0, 0, \pm 1)$ has horizontal T.P.

fig 13.79

Tangent plane of $z = f(x, y)$

$$F(x, y, z) = f(x, y) - z = 0$$

$$\nabla F = \langle f_x, f_y, -1 \rangle \text{ at } (a, b, f(a, b))$$

$$f_x(a, b)(x-a) + f_y(a, b)(y-b) - (z-f(a, b)) = 0$$

Therefore : $z = f_x(a, b)(x-a) + f_y(a, b)(y-b) + f(a, b)$

Ex. $z = f(x, y) = 32 - 3x^2 - 4y^2$ at $(2, 1, 16)$

$$z = f_x(2, 1)(x-2) + f_y(2, 1)(y-1) + f(2, 1)$$

$$= (-12)(x-2) + (-8)(y-1) + 16$$

$$= -12x - 8y + 48$$

fig 13.80

Linear approximation

The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane given by

$$L(x, y) = f_x(a, b)(x-a) + f_y(a, b)(y-b) + f(a, b)$$

Note: $L_x = f_x(a, b)$, $L_y = f_y(a, b)$, $L(a, b) = f(a, b)$

Ex.

Linear approximation to

$$f(x, y) = \frac{5}{x^2 + y^2} \quad \text{at } (-1, 2, 1)$$

$$f_x = -\frac{10x}{(x^2 + y^2)^2}, \quad f_y = -\frac{10y}{(x^2 + y^2)^2}$$

$$f_x(-1, 2) = -\frac{10(-1)}{25} = \frac{2}{5}, \quad f_y(-1, 2) = -\frac{10(2)}{25} = -\frac{4}{5}$$

$$\begin{aligned} L(x, y) &= \frac{2}{5}(x+1) - \frac{4}{5}(y-2) + 1 \\ &= \frac{2}{5}x - \frac{4}{5}y + 3 \end{aligned}$$

$$\begin{aligned} f(-1.05, 2.1) &\approx L(-1.05, 2.1) \\ &= 0.4(-0.05) - 0.8(0.1) + 1 \\ &= 0.9 \end{aligned}$$

$$f(-1.05, 2.1) \approx 0.907 \quad \left. \vphantom{f(-1.05, 2.1)} \right\} \text{good approximation.}$$

Differentials and Change

$z = f(x, y)$ approximated by $L(x, y)$ at $(a, b, f(a, b))$

$$f(x, y) \approx L(x, y) = f_x(a, b)(x-a) + f_y(a, b)(y-b) + f(a, b)$$

Exact change between points (x, y) and (a, b)

$$\Delta z = f(x, y) - f(a, b)$$

approximated by

$$\Delta z \approx L(x, y) - f(a, b)$$

$$= f_x(a, b)(x-a) + f_y(a, b)(y-b)$$

$$dz = L(x, y) - f(a, b)$$

$$= f_x(a, b) dx + f_y(a, b) dy$$

$$dx = x - a \quad \text{and} \quad dy = y - b$$

$$\Delta z \approx dz = f_x(a, b) dx + f_y(a, b) dy$$

The approximation improves as Δx and Δy approach 0.

Differentiability: $\Delta Z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$
 with $(\varepsilon_1, \varepsilon_2) \rightarrow (0,0)$ as $(\Delta x, \Delta y) \rightarrow (0,0)$
 $\Delta Z = dZ + \varepsilon_1\Delta x + \varepsilon_2\Delta y$
 Better approximation for smaller $\Delta x, \Delta y$.

Ex. $Z = f(x,y) = \frac{5}{x^2+y^2}$
 ΔZ from $(-1, 2)$ to $(-0.93, 1.94)$

$$f_x = -\frac{10x}{(x^2+y^2)^2} = \frac{2}{5} \text{ at } (-1, 2)$$

$$f_y = -\frac{10y}{(x^2+y^2)^2} = -\frac{4}{5} \text{ at } (-1, 2)$$

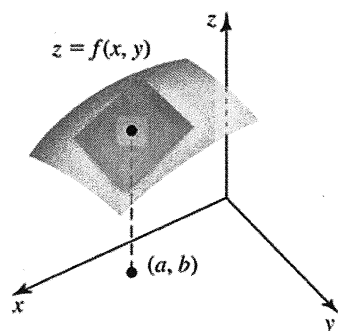
$$dx = 0.07, \quad dy = -0.06$$

$$\Delta Z \approx dZ = \frac{2}{5}(0.07) - \frac{4}{5}(-0.06) = 0.076$$

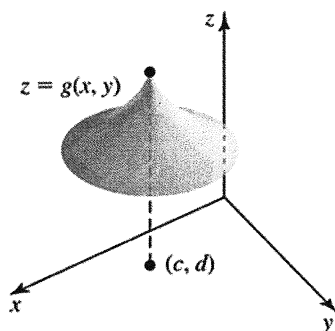
$$\Delta Z \approx 0.080$$

} good approxi.

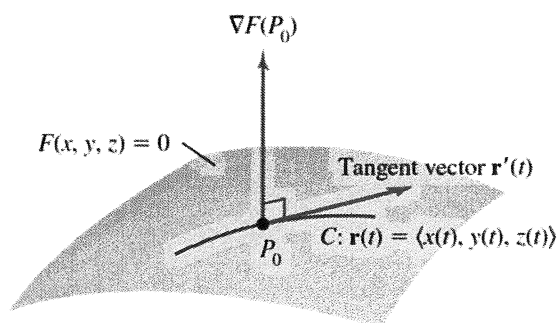
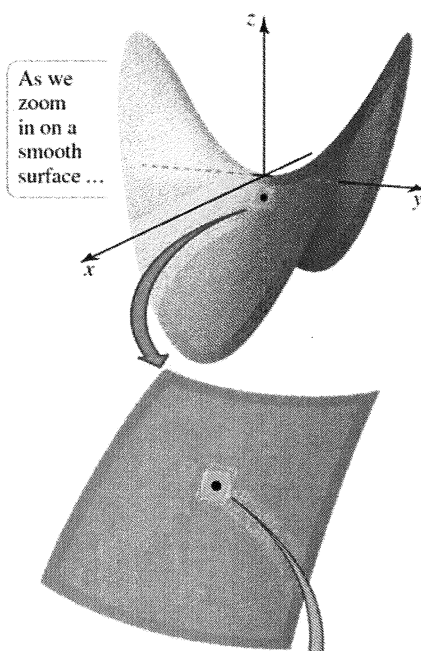
13.7

13.76

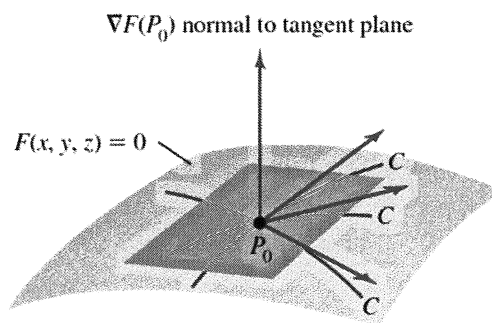
f differentiable at
 $(a, b) \Rightarrow$ tangent
 plane at $(a, b, f(a, b))$



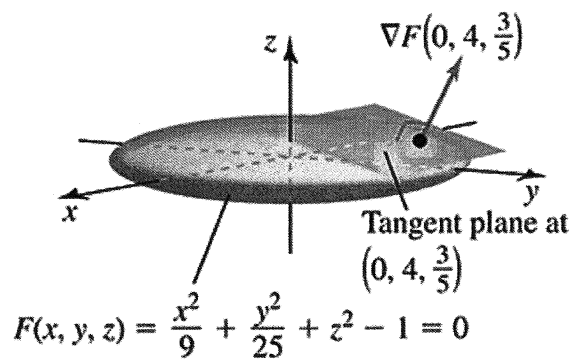
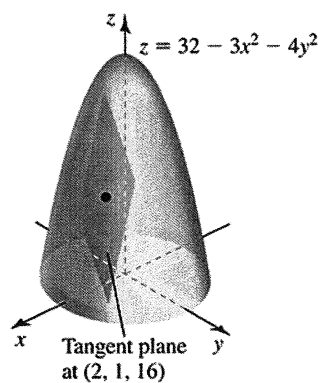
g not differentiable at
 $(c, d) \Rightarrow$ no tangent
 plane at $(c, d, g(c, d))$

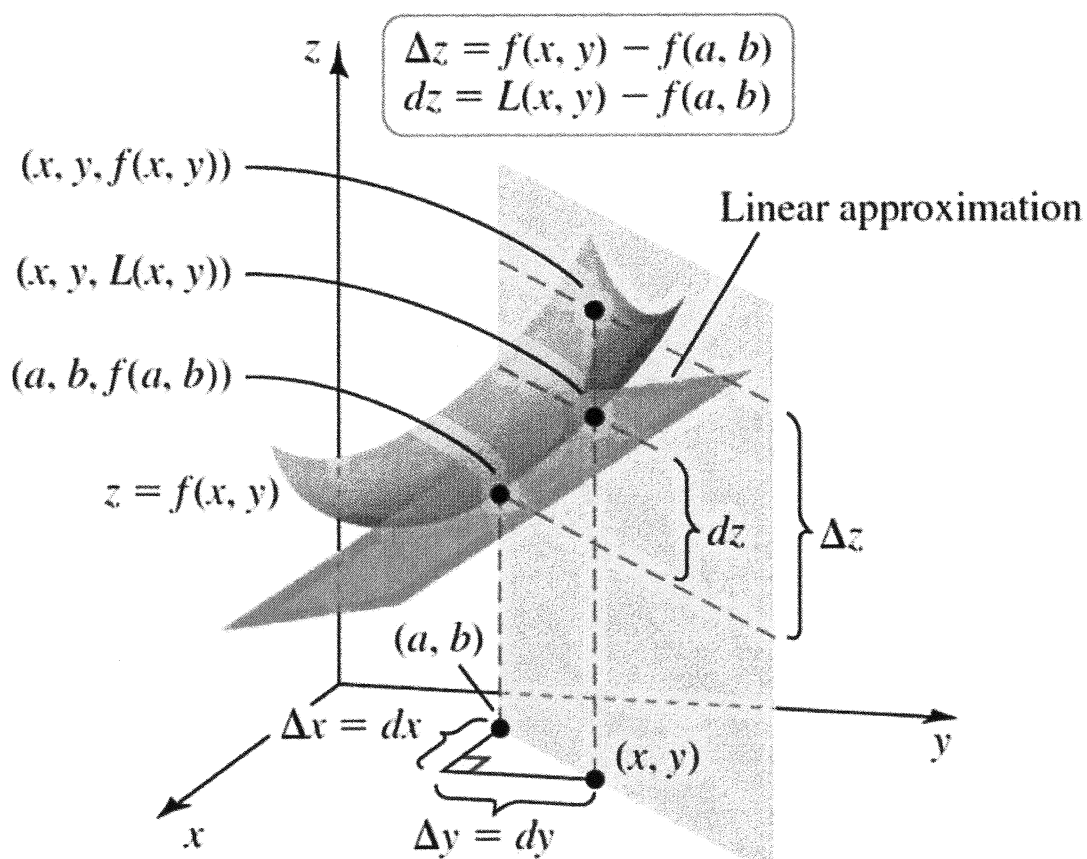
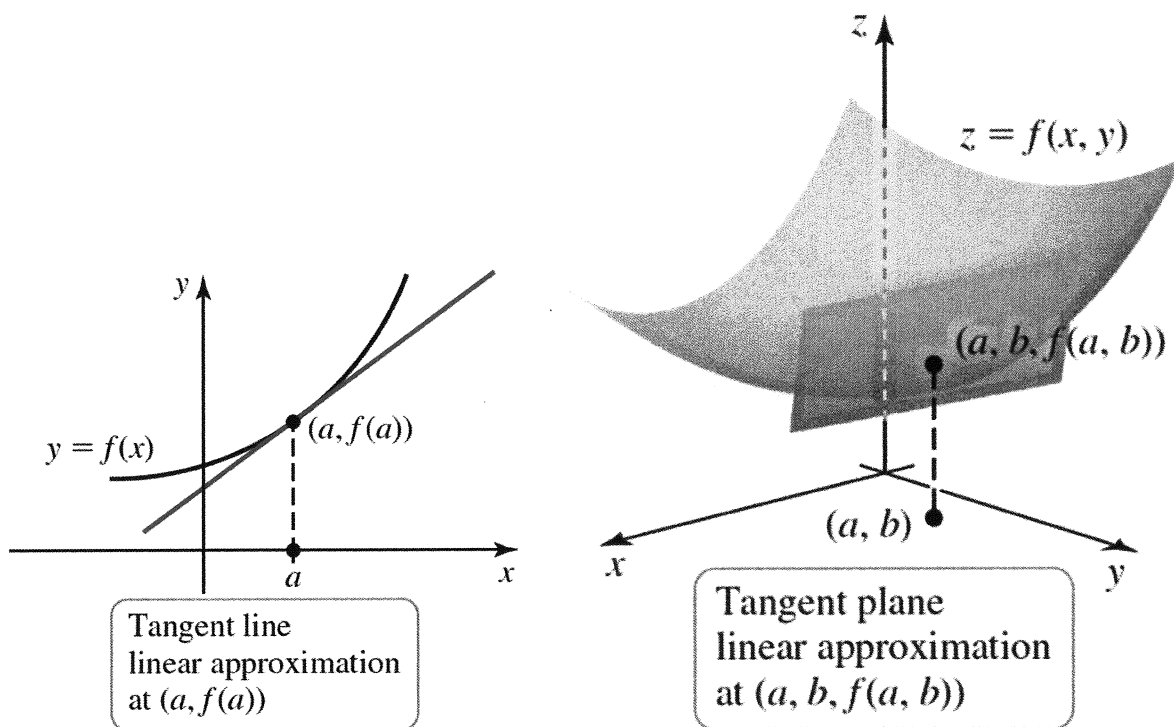
13.77

Vector tangent to C
 at P_0 is orthogonal
 to $\nabla F(P_0)$.



Tangent plane formed
 by tangent vectors for
 all curves C on the
 surface passing through P_0

13.7813.7913.80



13.83