

13.8 Maximum / minimum problems

See fig 13.85 for an illustration of various extrema : Local / absolute maximum / minimum.

Definition : Local maximum / minimum values

A function f has a local maximum value at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b) .

Note : ① In the domain of f , of course!
② Open disk : a small vicinity $\rightarrow$ local.

local minimum : $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b) .

The comparison between $f(x, y)$ and $f(a, b)$ is in a small neighborhood. $\rightarrow$ Local extrema

Theorem : If f has a local maximum or minimum at (a, b) and f_x and f_y exist at (a, b) , then $f_x(a, b) = f_y(a, b) = 0$

Proof : $g(x) = f(x, \underline{b})$ with $y = b$ fixed.
Local extremum at $(a, b) \rightarrow g'(a) = 0$
 $g'(a) = f_x(a, b) = 0$ *
 $h(y) = f(\underline{a}, y)$ with $x = a$ fixed.
Local extremum at $(a, b) \rightarrow h'(b) = 0$
 $h'(b) = f_y(a, b) = 0$ *

The above theorem provides candidates for local extrema. They are critical points.

Critical points

An interior point (a, b) in the domain of f is a Critical Point if either

- ① $f_x(a, b) = f_y(a, b) = 0$, or
- ② at least one of f_x and f_y does not exist at (a, b) .

Ex. $f(x, y) = xy(x-2)(y+3)$

Find all the C.P.

$$f_x = 2(x-1)y(y+3) = 0$$

$$f_y = x(x-2)(2y+3) = 0$$

$$f_x = 0 \rightarrow \text{① } x=1, \text{ ② } y=0, \text{ ③ } y=-3$$

$$\text{① } x=1, f_y = -(2y+3) = 0, y = (-\frac{3}{2})$$

$$\text{② } y=0, f_y = 3x(x-2) \rightarrow x=0 \text{ or } 2.$$

$$\text{③ } y=-3, f_y = -3x(x-2) \rightarrow x=0 \text{ or } 2$$

Five critical points in total:

$$(1, -\frac{3}{2}), (0, 0), (2, 0), (0, -3), (2, -3)$$

Some of them may correspond to Local Extrema.

Second derivative test.

Suppose that the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b) , where $f_x(a, b) = f_y(a, b) = 0$. Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$.

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b) .
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b) .
3. If $D(a, b) < 0$, then f has a saddle point at (a, b) .
4. If $D(a, b) = 0$, then the test is inconclusive.

What is a saddle point?

A critical point (a, b) is a saddle point if, in every open disk centered at (a, b) , there are points (x, y) for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.

Uphill in one direction from $(a, b, f(a, b))$ while downhill in another direction.

Consider $f(x, y) = x^2 - y^2$ at $(0, 0, 0)$.

Uphill in x direction while downhill in y direction, with $f_x(0, 0) = f_y(0, 0) = 0$

Ex. $f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$

$$f_x = 2x - 4, \quad f_y = 4y + 4$$

Critical point $(2, -1)$, just one.

$$f_{xx} = 2, \quad f_{yy} = 4, \quad f_{xy} = f_{yx} = 0$$

$$D = 8, \quad f_{xx} > 0 \rightarrow \text{A local minimum at } (2, -1).$$

$$f(x, y) = x^2 - 4x + 4 + 2y^2 + 4y + 2$$

$$= (x - 2)^2 + 2(y + 1)^2$$

Obviously a local minimum at $(2, -1)$
 $f(2, -1) = 0$.

Ex. $f(x, y) = xy(x - 2)(y + 3)$ studied before
 $(0, 0)$ is a critical point

$$f_{xx} = 2(y)(y + 3)$$

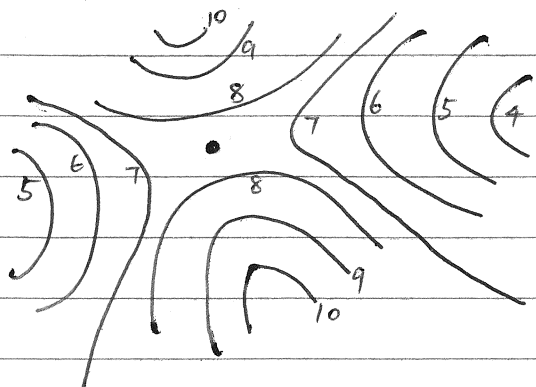
$$f_{yy} = 2(x)(x - 2)$$

$$f_{xy} = f_{yx} = (2x - 2)(2y + 3)$$

$$f_{xx} = f_{yy} = 0 \quad \text{and} \quad f_{xy} = f_{yx} = -6$$

$$D = -36 < 0. \quad \text{A Saddle Point.}$$

What is the characteristic of level curves near a saddle point?



Uphill in one direction while downhill in another direction!

Absolute maximum / minimum

DEFINITIONS Absolute Maximum/Minimum Values

If $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f , then f has an **absolute maximum value** at (a, b) . If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f , then f has an **absolute minimum value** at (a, b) .

PROCEDURE Finding Absolute Maximum/Minimum Values on Closed, Bounded Sets

Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R :

1. Determine the values of f at all critical points in R .
2. Find the maximum and minimum values of f on the boundary of R .
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R , and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R .

Ex. $f(x, y) = x^2 + y^2 - 2x + 2y + 5$ on
 $R = \{(x, y) : x^2 + y^2 \leq 4\}$
 R is closed and bounded.

Critical points from $f_x = 2x - 2 = 0$
 $f_y = 2y + 2 = 0$

$(1, -1)$ is in R $f(1, -1) = 3$

On the boundary of R

$$x^2 + y^2 = 4$$

$$x = 2 \cos t, \quad y = 2 \sin t$$

Parametrization

$$\begin{aligned} f(x, y) \rightarrow g(t) &= 4 \cos^2 t + 4 \sin^2 t \\ &\quad - 4 \cos t + 4 \sin t + 5 \\ &= 9 - 4 \cos t + 4 \sin t \end{aligned}$$

$$\frac{d}{dt} g(t) = 4 \sin t + 4 \cos t = 0$$

$$g'(t) = 0 \quad \text{if} \quad \tan t = -1 : t = -\frac{\pi}{4} \text{ and } \frac{3\pi}{4}$$

$$(x = \sqrt{2}, y = -\sqrt{2})$$

$$(x = -\sqrt{2}, y = \sqrt{2})$$

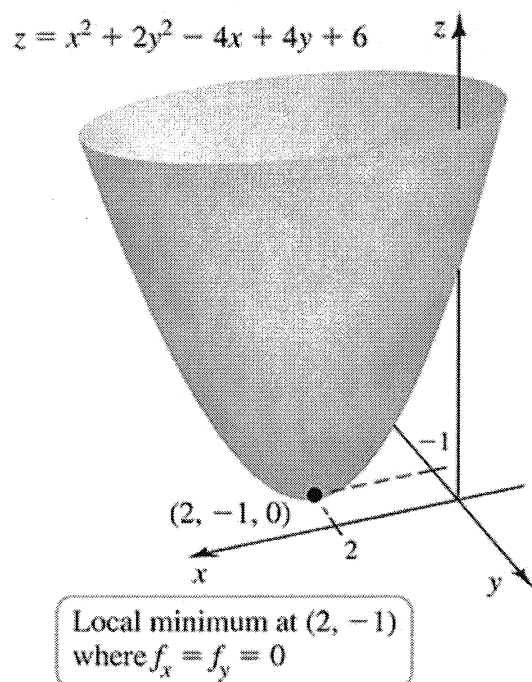
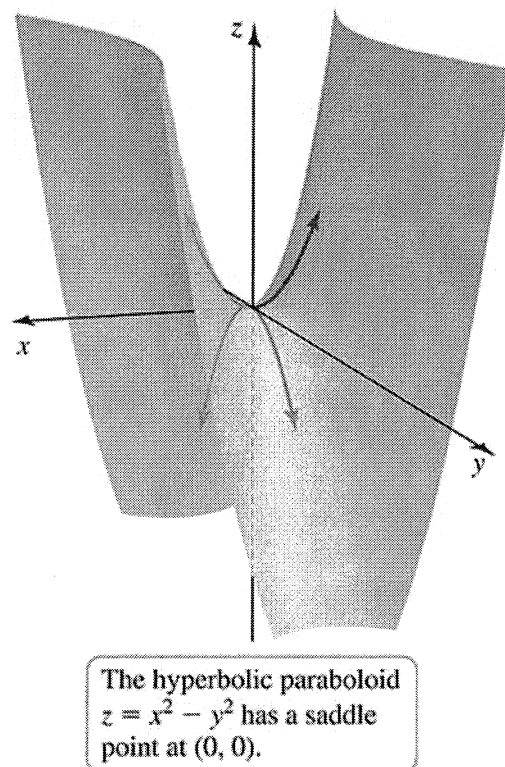
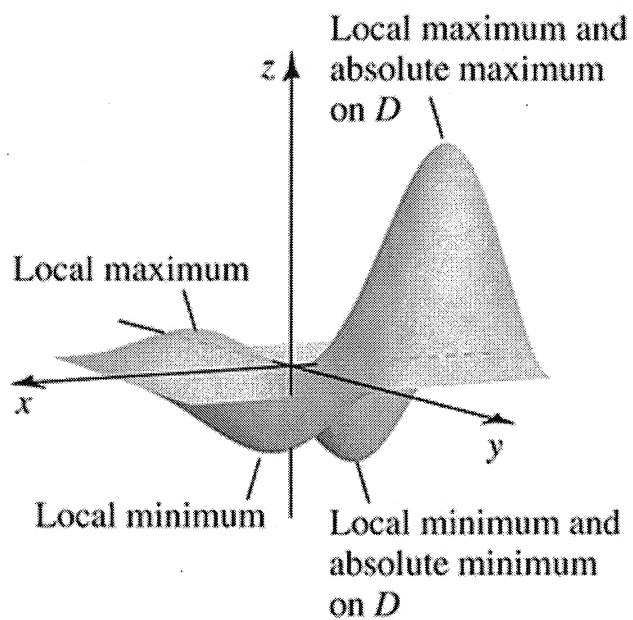
$$f(\sqrt{2}, -\sqrt{2}) = 9 - 2x + 2y = 9 - 4\sqrt{2}$$

$$f(-\sqrt{2}, \sqrt{2}) = 9 - 2x + 2y = 9 + 4\sqrt{2}$$

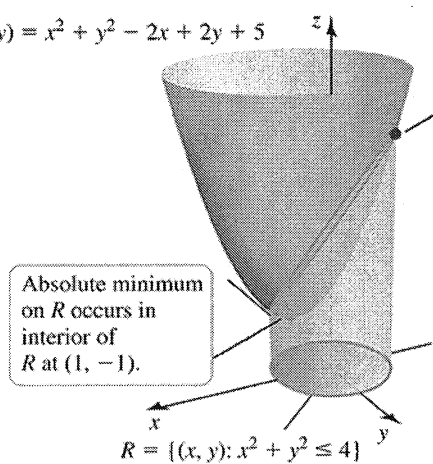
Compare $f(1, -1) = 3 \rightarrow \text{abs. minimum}$

$$f(\sqrt{2}, -\sqrt{2}) = 9 - 4\sqrt{2} \approx 3.3 \rightarrow ?$$

$$f(-\sqrt{2}, \sqrt{2}) = 9 + 4\sqrt{2} \rightarrow \text{abs. maximum}$$



$$f(x, y) = x^2 + y^2 - 2x + 2y + 5$$



Absolute maximum on R occurs on boundary of R at $(-\sqrt{2}, \sqrt{2})$.

Maximum value of f on boundary of R occurs at $(-\sqrt{2}, \sqrt{2})$, where $\theta = -\frac{3\pi}{4}$.

