

14 Multiple Integration

14.1 Double integrals over rectangular regions

Volumes of solids

Consider a 3D SOLID bounded by $z = f(x, y)$ and R (a region in the xy plane, a rectangular region).
here in 14.1

Approximate the solid by a collection of boxes.
See fig 14.1

① Assume $z = f(x, y) \geq 0$ on $R = \{(x, y) : x \in [a, b], y \in [c, d]\}$
 "a rectangle" for R .

② A partition of R : R divided into n rectangular subregions using lines parallel to x and y axes.

area of k th subregion $= \Delta x_k \Delta y_k$
 (x_k^*, y_k^*) : Any point in k th subregion.
 $1 \leq k \leq n$.

③ k th box: volume $= f(x_k^*, y_k^*) \Delta x_k \Delta y_k$
 Sum:

Approximation to the volume of the solid

$$V \approx \sum_{k=1}^n f(x_k^*, y_k^*) \Delta x_k \Delta y_k$$

④ Let $\Delta A_k = \Delta x_k \Delta y_k$ all approach zero.
 $\Delta A_k \rightarrow 0$ for $1 \leq k \leq n$ with $n \rightarrow \infty$.

Limit of the Riemann sum = Volume of the solid.

See figs 14.2, 14.3, 14.4

If the limit exists for all partitions of R and for all choices of (x_k^*, y_k^*) within those partitions, then the double integral of f over R is defined and given by

$$\iint_R f(x, y) dA = \lim_{n \rightarrow \infty} \underbrace{\sum_{k=1}^n}_{\text{iii)}} \underbrace{f(x_k^*, y_k^*)}_{\text{ii)}} \underbrace{\Delta A_k}_{\text{i)}} \quad \text{iv)}$$

- i) Partition of R ii) Volume of a box
iii) Sum iv) Limit

Now, how to compute the double integral?

$$z = f(x, y) = 6 - 2x - y \quad \text{for } R = \{(x, y) : x \in [0, 1], y \in [0, 2]\}$$

$$V = \iint_R f(x, y) dA = \iint_R (6 - 2x - y) dA$$

Iterated integrals

$$V = \int_0^1 A(x) dx \quad \text{with } A(x) = \int_0^2 (6 - 2x - y) dy$$

for a FIXED value of x .

$$V = \int_0^1 A(x) dx = \int_0^1 \left[\int_0^2 (6 - 2x - y) dy \right] dx$$

$A(x)$ for fixed x .

$$= \int_0^1 \left(6y - 2xy - \frac{y^2}{2} \right) \Big|_0^2 dx$$

$$= \int_0^1 (12 - 4x - 2) dx = (10x - 2x^2) \Big|_0^1 = 8$$

$$A(x) = 10 - 4x$$

See fig 14.5.

Let's change the order in the iterated integral.

$$V = \int_0^2 A(y) dy \quad \text{with} \quad A(y) = \int_0^1 (6-2x-y) dx$$

for a FIXED value of y .

$$\begin{aligned} V &= \int_0^2 A(y) dy = \int_0^2 \left[\int_0^1 (6-2x-y) dx \right] dy \\ &= \int_0^2 (6x - x^2 - xy) \Big|_0^1 dy \\ &= \int_0^2 (6 - 1 - y) dy = \left(5y - \frac{y^2}{2} \right) \Big|_0^2 = 10 - 2 = 8 \end{aligned}$$

See fig 14.6.

Geometrically, $\underline{A(x)} = \int_c^d f(x,y) dy$

is the area of a slice at a fixed value of $\underline{x}$.

$$\underline{A(y)} = \int_a^b f(x,y) dx$$

is the area of a slice at a fixed value of $\underline{y}$

$$R = \{ (x,y) : a \leq x \leq b, c \leq y \leq d \}$$

$$\begin{aligned} \iint_R f(x,y) dA &= \int_c^d \int_a^b f(x,y) dx dy \\ &= \int_a^b \int_c^d f(x,y) dy dx \end{aligned}$$

$$z = 4 + 9x^2y^2, \quad R = \{(x, y) : x \in [-1, 1], y \in [0, 2]\}$$

$$\begin{aligned} & \iint_R f(x, y) dA \\ &= \int_0^2 \int_{-1}^1 (4 + 9x^2y^2) dx dy \\ &= \int_0^2 (4x + 3x^3y^2) \Big|_{-1}^1 dy \\ &= \int_0^2 (8 + 6y^2) dy = (8y + 2y^3) \Big|_0^2 \\ &= 16 + 16 = 32 \end{aligned}$$

Here $A(y) = 8 + 6y^2$ is computed first.

$$\begin{aligned} \text{Alternatively, } & \iint_R f(x, y) dA \\ &= \int_{-1}^1 \int_0^2 (4 + 9x^2y^2) dy dx \\ &= \int_{-1}^1 (4y + 3x^2y^3) \Big|_0^2 dx \\ &= \int_{-1}^1 (8 + 24x^2) dx \\ &= (8x + 8x^3) \Big|_{-1}^1 = 16 + 16 = 32 \end{aligned}$$

Here $A(x) = 8 + 24x^2$ is computed first.

Two iterated integrals agree! Fubini's theorem

Choosing a convenient order in iterated integrals

$$\begin{aligned}
 & \iint_R y e^{xy} dA, \quad R: x \in [0, 1], y \in [0, \ln 2] \\
 &= \int_0^{\ln 2} \int_0^1 y e^{xy} dx dy \\
 &= \int_0^{\ln 2} e^{xy} \Big|_{x=0}^1 dy \\
 &= \int_0^{\ln 2} (e^y - 1) dy = (e^y - y) \Big|_0^{\ln 2} \\
 &= (2 - \ln 2) - 1 = 1 - \ln 2
 \end{aligned}$$

Here $A(y) = \int_0^1 y e^{xy} dx$

$$= e^{xy} \Big|_0^1 = e^y - 1 \quad \text{is computed first.}$$

The other order involves integration by parts, thus more complicated.

Average value $\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x, y) dA$

$$f(x, y) = 2 - x - y, \quad R: x \in [0, 2], y \in [0, 2]$$

$$\begin{aligned}
 \iint_R f(x, y) dA &= \int_0^2 \int_0^2 (2 - x - y) dx dy \\
 &= \int_0^2 \left(2x - \frac{x^2}{2} - xy \right) \Big|_0^2 dy
 \end{aligned}$$

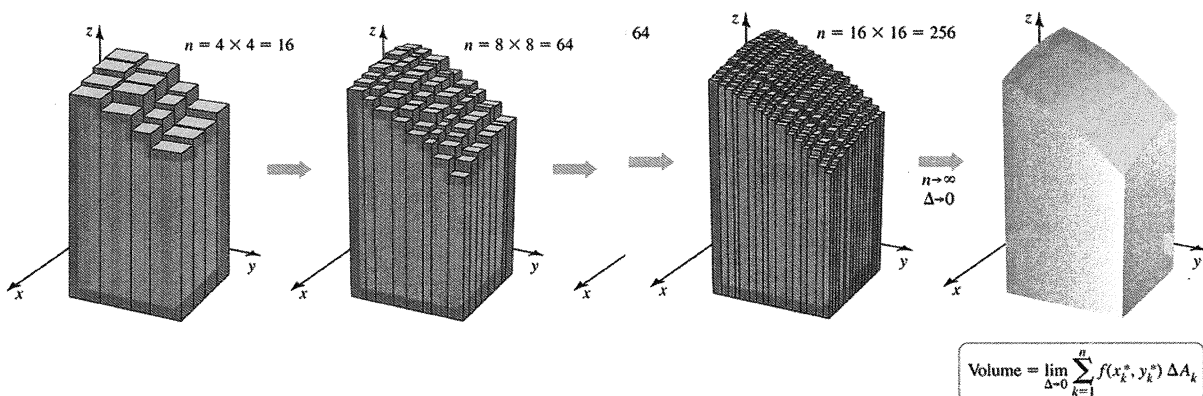
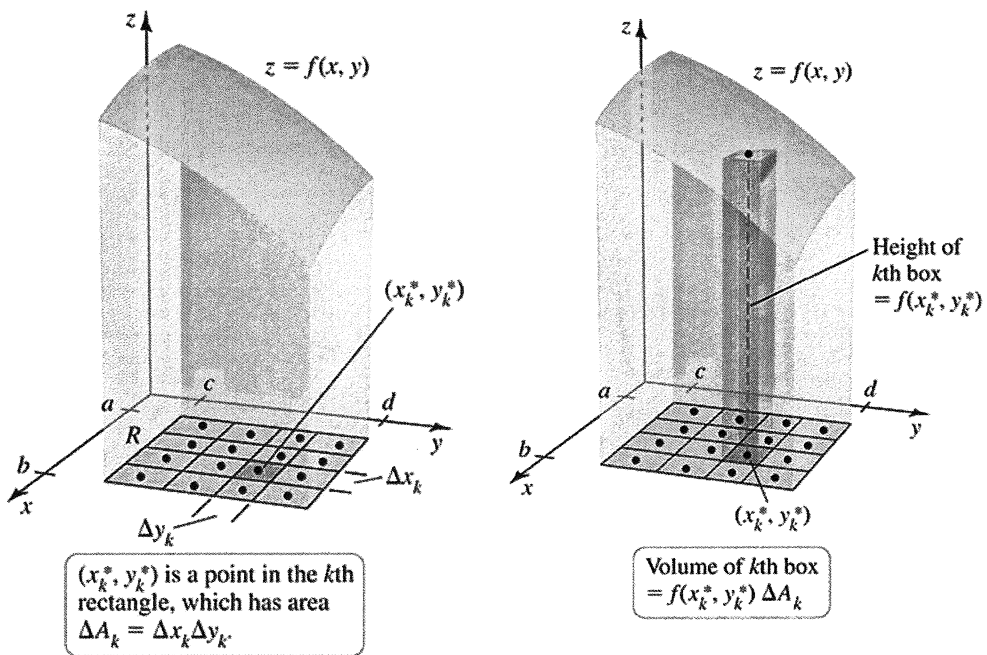
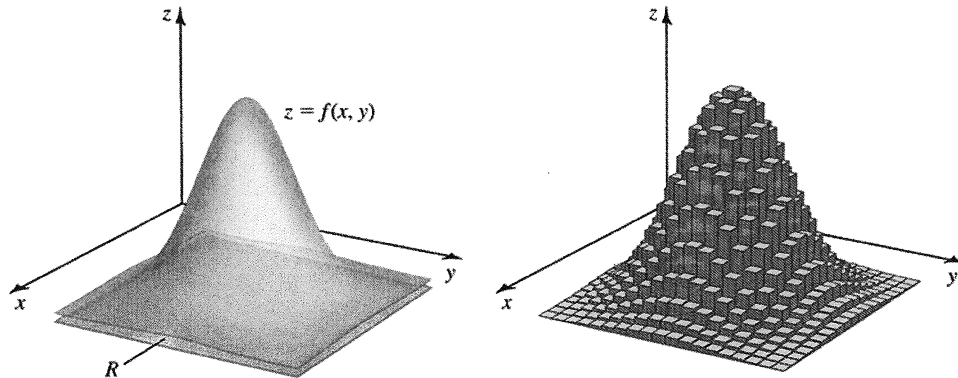
$$= \int_0^2 \left(4 - 2 - 2y \right) dy$$

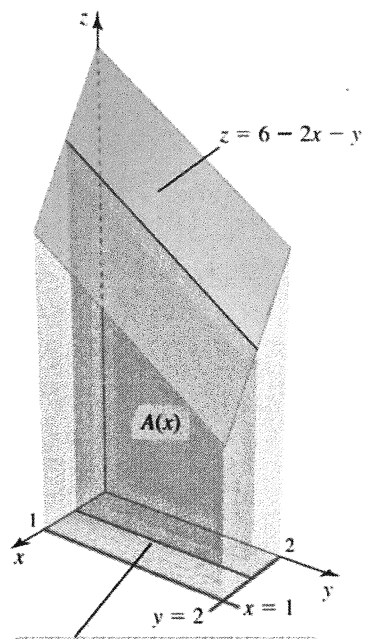
$$= \int_0^2 (2 - 2y) dy = (2y - y^2) \Big|_0^2 = 0$$

$$\bar{f} = \frac{1}{4} (0) = 0$$

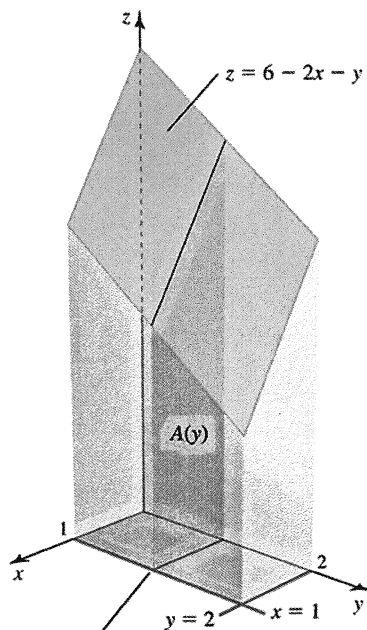
$f(0, 0) = 2$, $f(2, 0) = f(0, 2) = 0$, $f(2, 2) = -2$
Simple average = 0 as well: linearity.

A three-dimensional solid bounded by $z = f(x, y)$ and a region R in the xy -plane is approximated by a collection of boxes.





A slice at a fixed value of x has area $A(x)$, where $0 \leq x \leq 1$.



A slice at a fixed value of y has area $A(y)$, where $0 \leq y \leq 2$.