

14.2. Double integrals over general regions.

A general region R in xy plane:

A Closed & Bounded Region

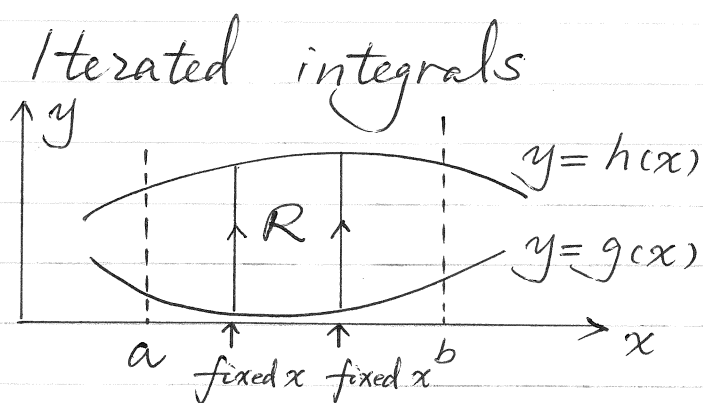
- A partition consisting of rectangles does NOT cover R exactly, however. fig 9

$$V \approx \sum_{k=1}^n f(x_k^*, y_k^*) \Delta A_k, \quad \text{sum over } n \text{ rectangles}$$

Δ : maximum length of the diagonals of the n rectangles that lie entirely within R .

$$\iint_R f(x, y) dA = \lim_{\substack{\Delta \rightarrow 0 \\ (n \rightarrow \infty)}} \sum_{k=1}^n f(x_k^*, y_k^*) \Delta A_k$$

Again, partition, sum, and limit, as for rectangular regions discussed in 14.1.



$$\iint_R f(x, y) dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y) dy dx$$

in which $\int_{g(x)}^{h(x)} f(x, y) dy = A(x)$ for fixed x
is computed first.

$$A(x) = \int_{g(x)}^{h(x)} \underline{f(x, y)} \underline{dy}, \quad \text{y going from } g(x) \text{ to } h(x).$$

(x fixed)

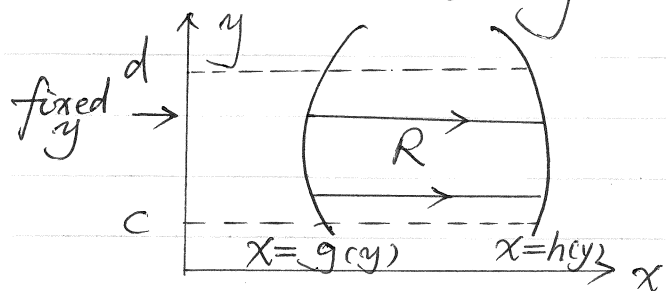
Ex $f(x,y) = 2x^2y$, R is bounded by $y = 3x^2$ and $y = 16 - x^2$

$$\begin{cases} y = 3x^2 \\ y = 16 - x^2 \end{cases} \rightarrow x = \pm 2$$

$$a = -2, \quad b = +2, \quad g(x) = 3x^2, \quad h(x) = 16 - x^2$$

$$\begin{aligned} \iint_R f(x,y) dA &= \int_{-2}^2 \int_{3x^2}^{16-x^2} 2x^2y \, dy \, dx \\ &= \int_{-2}^2 x^2 y^2 \Big|_{y=3x^2}^{16-x^2} dx \\ &= \int_{-2}^2 (256x^2 - 32x^4 - 8x^6) dx \\ &\approx 663 \end{aligned}$$

Now, let's change the perspective

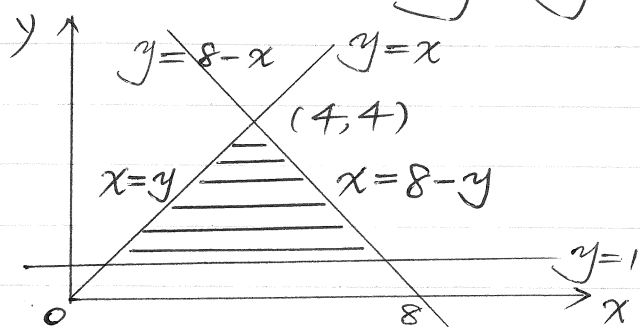


$$\begin{aligned} \iint_R f(x,y) dA &= \int_c^d \int_{g(y)}^{h(y)} f(x,y) dx dy \\ &= \int_c^d \int_{g(y)}^{h(y)} f(x,y) dx dy \end{aligned}$$

in which $\int_{g(y)}^{h(y)} f(x,y) dx = A(y)$ for fixed y is computed first.

$$A(y) = \int_{g(y)}^{h(y)} \underbrace{f(x,y)}_{\text{fixed}} dx, \quad x \text{ running from } g(y) \text{ to } h(y).$$

Ex $f(x, y) = 2 + \frac{1}{y}$,
 R bounded by $y = x$, $y = 8 - x$, $y = 1$



Left: $x = y$
 Right: $x = 8 - y$

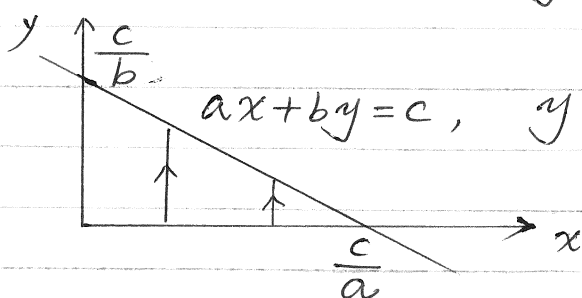
$$\begin{aligned}
 c = 1, \quad d = 4, \quad x = g(y) = y, \quad x = h(y) = 8 - y \\
 \int_1^4 \int_y^{8-y} \left(2 + \frac{1}{y}\right) dx dy &= \int_1^4 \left(2x + \frac{x}{y}\right) \Big|_{x=y}^{8-y} dy \\
 &= \int_1^4 \left(2(8-2y) + \frac{1}{y}(8-2y)\right) dy \\
 &= \int_1^4 \left(2 + \frac{1}{y}\right)(8-2y) dy = 12 + 8 \ln 4
 \end{aligned}$$

More exercises

Volume of a tetrahedron (Pyramid with four triangular faces)

Bounded by the plane $z = c - ax - by$, and the three coordinate planes $x = 0$, $y = 0$, $z = 0$ in the 1st octant.

Region R bounded by $ax + by = c$, $x = 0$, $y = 0$

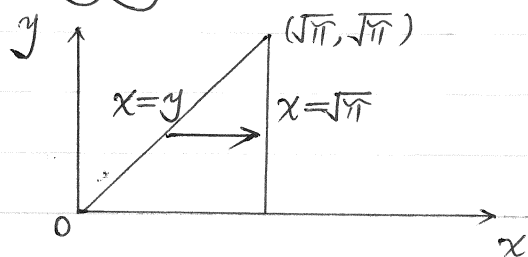


$$\begin{aligned}
 \iint_R z dx dy \\
 = \int_0^{\frac{c}{a}} \int_0^{\frac{c-ax}{b}} (c - ax - by) dy dx
 \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\frac{c}{a}} \left(cy - axy - \frac{b}{2} y^2 \right) \bigg|_{y=0}^{\frac{c-ax}{b}} dx \\
&= \int_0^{\frac{c}{a}} \frac{1}{2b} (c-ax)^2 dx \\
&= \frac{1}{6ab} (ax-c)^3 \bigg|_0^{\frac{c}{a}} = \frac{c^3}{6ab} = \frac{1}{3} \left(\frac{c^2}{2ab} \right) c
\end{aligned}$$

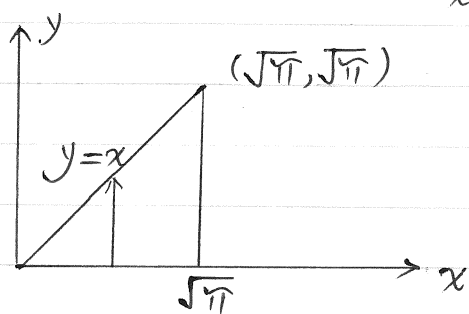
$\downarrow$ base area $\downarrow$ height.

Changing the order



$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin x^2 dx dy$$

This order can not be executed, however.



$$\int_0^{\sqrt{\pi}} \int_0^x \sin x^2 dy dx$$

$$= \int_0^{\sqrt{\pi}} x \sin x^2 dx$$

$$= -\frac{1}{2} \cos x^2 \bigg|_0^{\sqrt{\pi}} = -\frac{1}{2} (-2) = 1$$

Just change the way of looking at the region R.

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} f(x,y) dx dy = \int_0^{\sqrt{\pi}} \int_0^x f(x,y) dy dx$$

Understand what R is from the left, and then change the order to get the right to proceed.

Regions between 2 surfaces.

Find the volume of the solid bounded by $z = x^2 + y^2$ from below and $z = 8 - (x^2 + y^2)$ from above.

Region R : $x^2 + y^2 = 8 - x^2 - y^2 \rightarrow x^2 + y^2 = 4$
enclosed by this circle.

$$\text{Volume} = \iint_R [8 - x^2 - y^2 - (x^2 + y^2)] dA$$

$$= 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (8 - 2x^2 - 2y^2) dy dx$$

↓
the double integral is computed only in the first quadrant, $\frac{1}{4}$ of R .

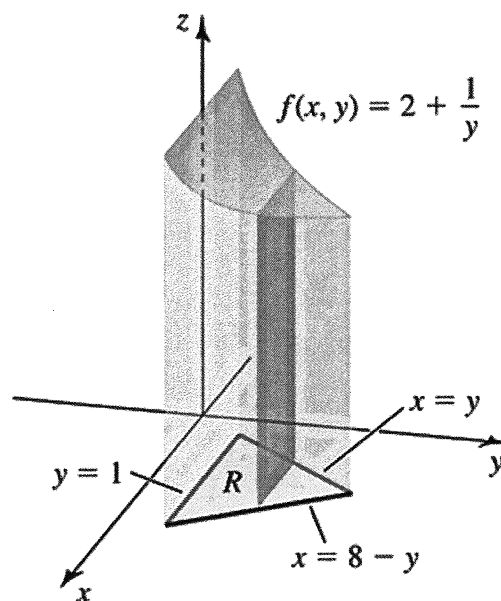
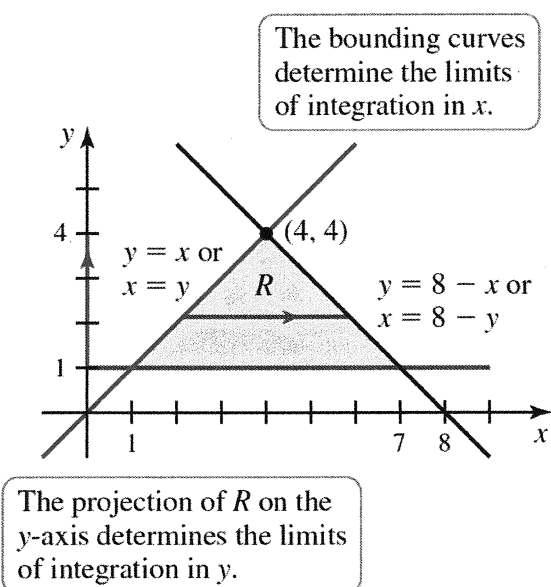
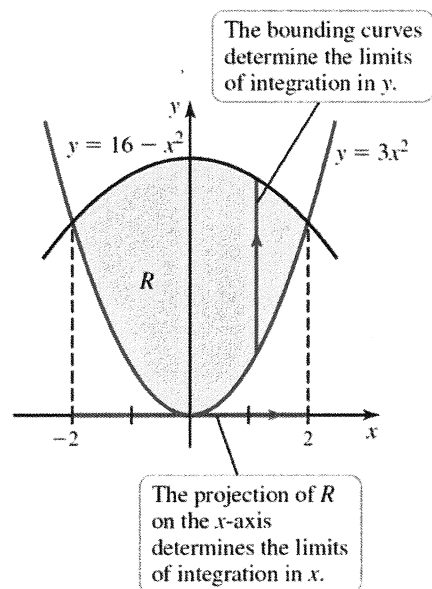
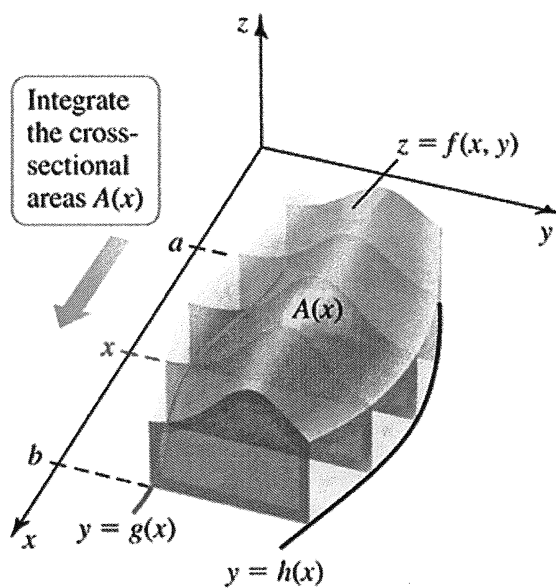
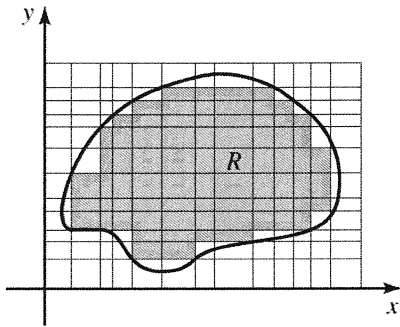
$$= 8 \int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2) dy dx$$

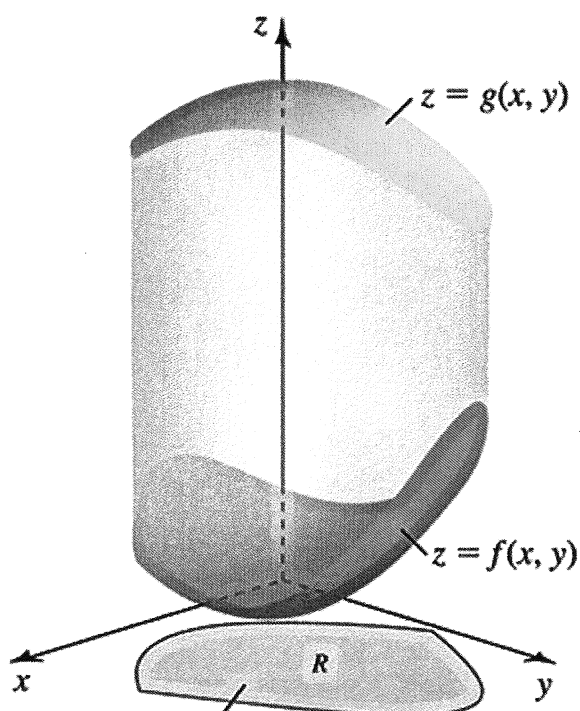
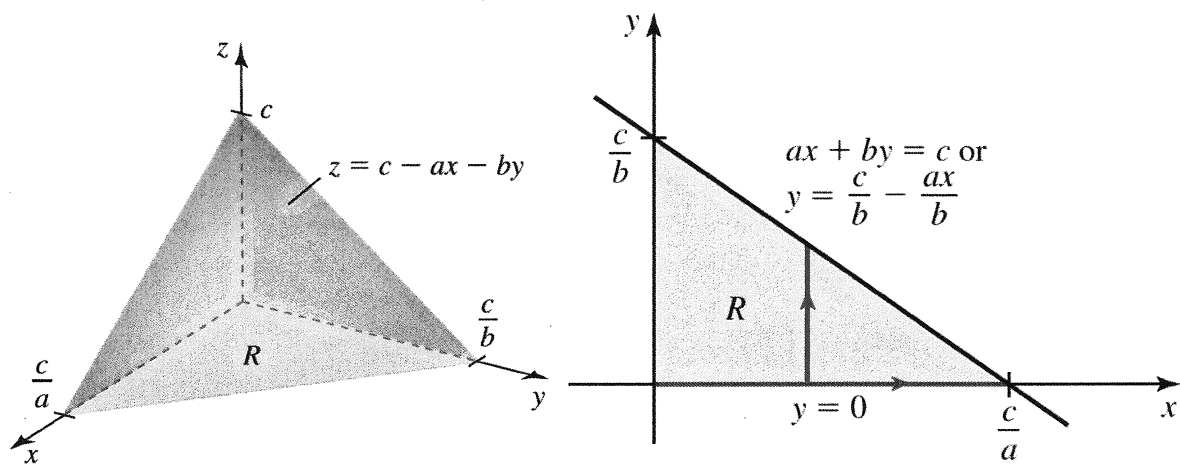
$$= 8 \times \frac{2}{3} \int_0^2 (4 - x^2)^{\frac{3}{2}} dx$$

$$x = 2 \sin \theta$$

Substitution

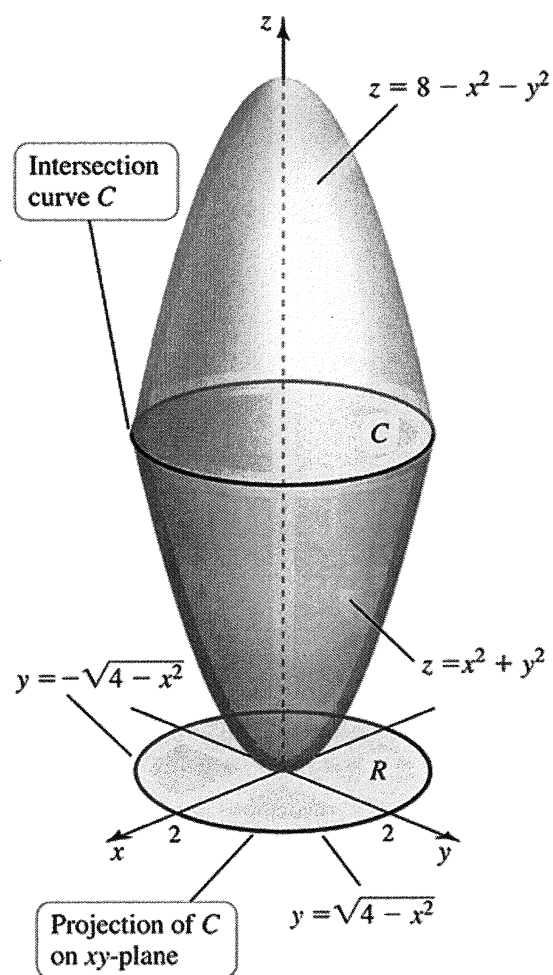
$$= \frac{16}{3} \times 16 \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta = 16\pi$$





Shadow of the solid
in the xy -plane

$$\text{Volume} = \iint_R (g(x, y) - f(x, y)) dA$$



Intersection
curve C

Projection of C
on xy -plane