

14.3 Double integrals in polar coordinates

1. To formulate double integrals in polar coordinates
2. To change double integrals from Cartesian coordinates to polar coordinates

Polar rectangular region.

$$R = \{ (r, \theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta \}$$

See fig 14.28

for a few examples.

$[a, b]$ divided into M subintervals
of equal length $\frac{b-a}{M} = \Delta r$

$[\alpha, \beta]$ divided into m subintervals
of equal length $\frac{\beta-\alpha}{m} = \Delta \theta$

fig 14.29

The arcs and rays divide R into $n = Mm$ polar rectangles.

$$V \cong \sum_{k=1}^n f(r_k^*, \theta_k^*) \Delta A_k \quad : \text{an approximation}$$

$$\lim_{\substack{\Delta \rightarrow 0 \\ (n \rightarrow \infty)}} \sum_{k=1}^n f(r_k^*, \theta_k^*) \Delta A_k = \iint_R f(r, \theta) dA$$

Δ : maximum of Δr and $\Delta \theta$.
 $\Delta \rightarrow 0$, then $n \rightarrow \infty$.

$$\begin{aligned} \Delta A_k &= \frac{1}{2} \left(r_k^* + \frac{\Delta r}{2} \right)^2 \Delta \theta - \frac{1}{2} \left(r_k^* - \frac{\Delta r}{2} \right)^2 \Delta \theta \\ &= r_k^* \Delta r \Delta \theta \end{aligned}$$

fig 14.30

$$dA = r dr d\theta$$

$$\iint_R f(r, \theta) dA = \int_{\alpha}^{\beta} \int_a^b f(r, \theta) r dr d\theta$$

Ex. The volume enclosed by $z = 9 - x^2 - y^2$ and the xy plane.

$$z = 9 - r^2, \quad z = 0, \quad r^2 = 9, \quad r = 3$$

Polar region $R = \{(r, \theta) : r \in [0, 3], \theta \in [0, 2\pi]\}$

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^3 (9 - r^2) r \, dr \, d\theta \\ &= \int_0^{2\pi} \left(\frac{9}{2} r^2 - \frac{1}{4} r^4 \right) \Big|_0^3 d\theta = 2\pi \left(\frac{81}{2} - \frac{81}{4} \right) \\ &= \frac{81}{2} \pi \end{aligned}$$

fig 14.31

Ex. Volume bounded by $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$.

$$r^2 = 8 - r^2, \quad r = 2$$

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^2 [(8 - r^2) - r^2] r \, dr \, d\theta \\ &= \int_0^{2\pi} \left(4r^2 - \frac{r^4}{2} \right) \Big|_0^2 d\theta = 2\pi (16 - 8) \\ &= 16\pi \end{aligned}$$

fig 14.32

Ex. Volume beneath $z = xy + 10$ and above the annular region $R = \{(r, \theta) : r \in [2, 4], \theta \in [0, 2\pi]\}$

$$xy + 10 = r^2 \sin \theta \cos \theta + 10 = \frac{1}{2} r^2 \sin 2\theta + 10$$

$$\begin{aligned} V &= \int_0^{2\pi} \int_2^4 \left(\frac{1}{2} r^2 \sin 2\theta + 10 \right) r \, dr \, d\theta \\ &= \int_0^{2\pi} \left(\left(\frac{r^4}{8} \sin 2\theta + 5r^2 \right) \Big|_{r=2}^4 \right) d\theta \\ &= \int_0^{2\pi} (30 \sin 2\theta + 60) d\theta = 120\pi \end{aligned}$$

fig 14.33

Double integrals over more general regions

$$R = \{ (r, \theta) : \theta \in [\alpha, \beta], 0 \leq g(\theta) \leq r \leq h(\theta) \}$$

fig 14.34

$$\iint_R f(r, \theta) dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} f(r, \theta) r dr d\theta$$

inner along r , outer along θ .

Ex. Specifying regions: R is outside $r=2$ and inside $r=4\cos\theta$

$r=2$: circle centered at origin.

$$r=4\cos\theta: x^2+y^2=4x, \quad (x-2)^2+y^2=4$$

a circle as well.

$$2=4\cos\theta, \quad \cos\theta=\frac{1}{2}, \quad \theta=\pm\frac{\pi}{3} : \text{intersections}$$

fig 14.35

R needs to be divided into 3 sub-regions, each being a general region bounded by $\theta=\alpha$, $\theta=\beta$, $r=g(\theta)$, $r=h(\theta)$, as defined above

$$\begin{aligned} \iint_R f(r, \theta) dA &= \int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} \int_0^{4\cos\theta} f(r, \theta) r dr d\theta & R_1 \\ &+ \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_0^2 f(r, \theta) r dr d\theta & R_2 \\ &+ \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{4\cos\theta} f(r, \theta) r dr d\theta & R_3 \end{aligned}$$

R_1, R_2, R_3 : Three subregions of the same type.

Areas of polar regions

$$R = \{(r, \theta) : \alpha \leq \theta \leq \beta, 0 \leq g(\theta) \leq r \leq h(\theta)\}$$

$$\text{Area} = \iint_R dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} r \, dr \, d\theta$$

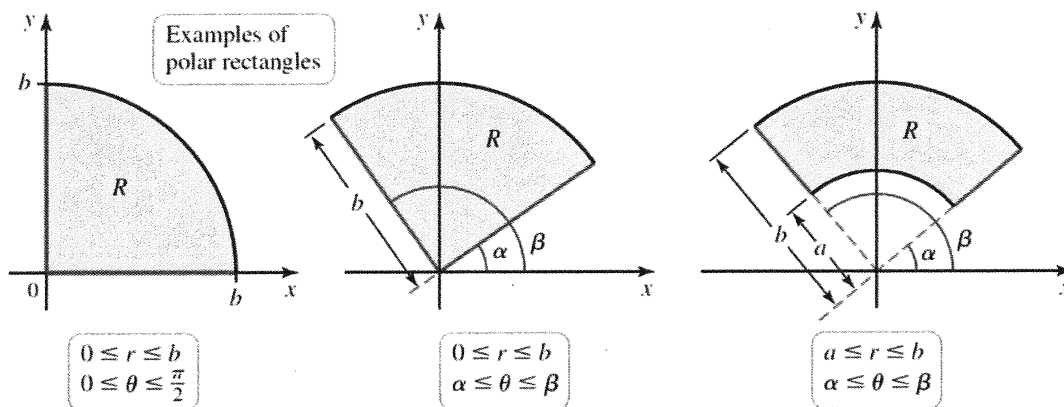
Ex. inner along r , outer along θ
 Area of the region in the 1st and 4th quadrants
 outside $r = \sqrt{2}$ and inside $r^2 = 4 \cos 2\theta$.
 (Lemniscate)

fig 14.37

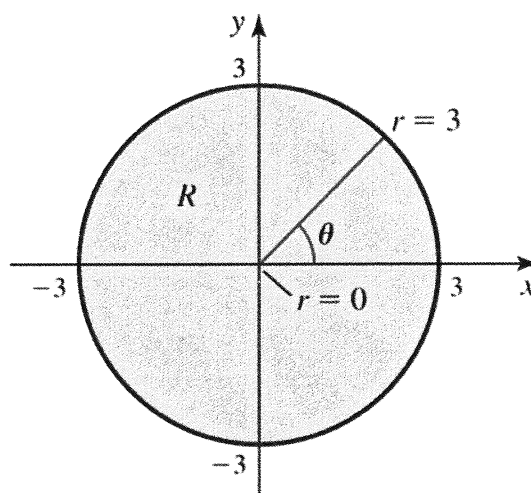
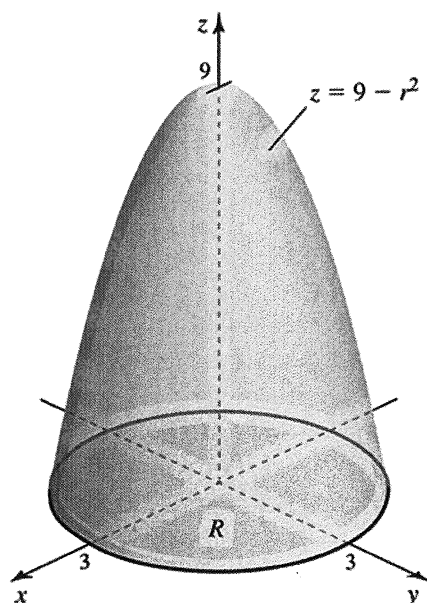
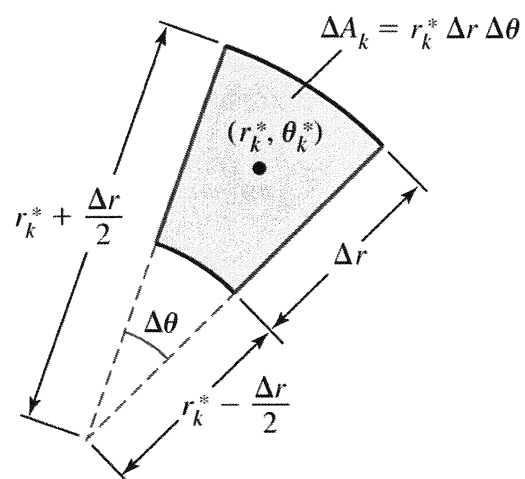
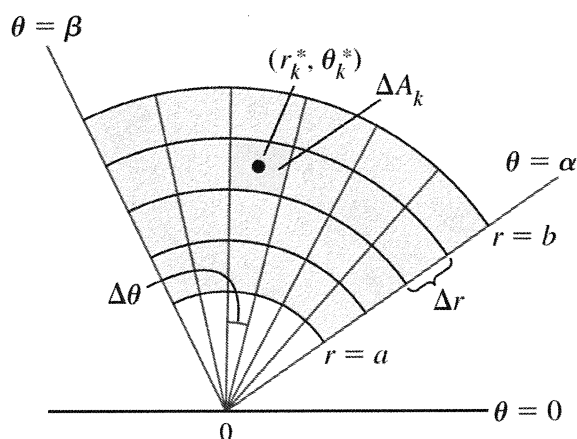
$$(\sqrt{2})^2 = 4 \cos 2\theta, \quad \cos 2\theta = \frac{1}{2}$$

$$\theta = \pm \frac{\pi}{6}$$

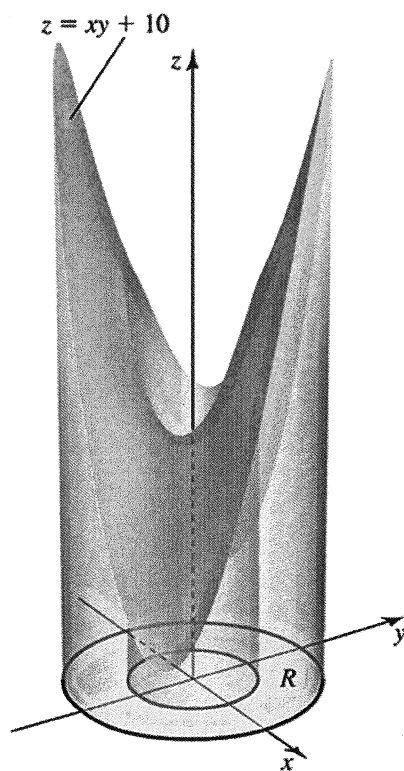
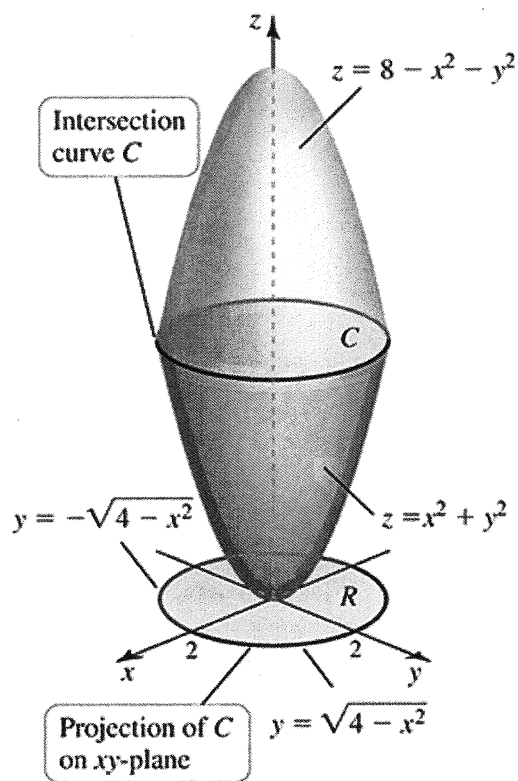
$$\begin{aligned} \text{Area} &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_{\sqrt{2}}^{2\sqrt{\cos 2\theta}} r \, dr \, d\theta \\ &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left. \frac{1}{2} r^2 \right|_{\sqrt{2}}^{2\sqrt{\cos 2\theta}} d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (2 \cos 2\theta - 1) d\theta \\ &= (\sin 2\theta - \theta) \Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \sqrt{3} - \frac{\pi}{3} \end{aligned}$$



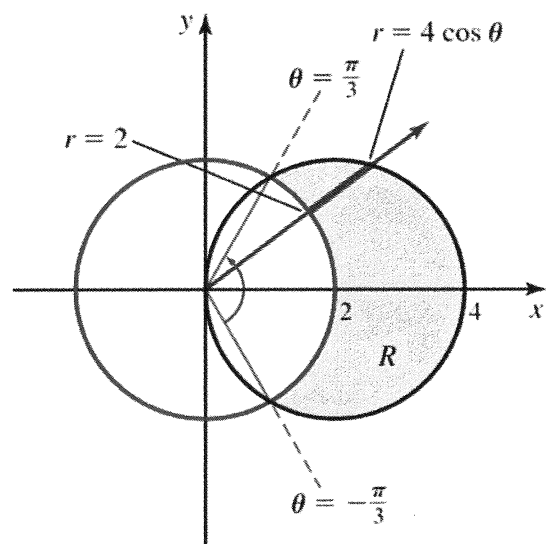
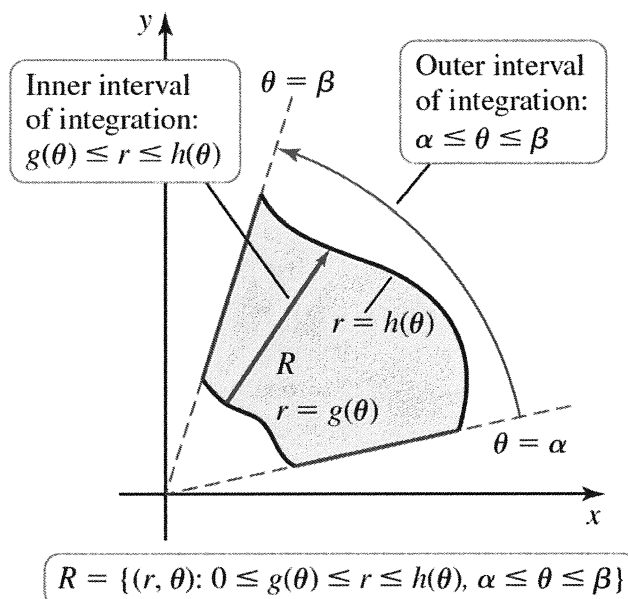
$$R = \{(r, \theta): a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$



$$R = \{(r, \theta): 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$$



Radial lines enter the region R at $r = 2$ and exit the region at $r = 4 \cos \theta$.



The inner and outer boundaries of R are traversed, for $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$.

