

14.4 Triple integrals

Consider $W = f(x, y, z)$ defined on a closed and bounded region D in $\mathbb{R}^3$.

- ① A partition of D by three sets of planes parallel to the xy , yz , xz planes, respectively.

D divided into small boxes ordered from $k=1$ to $k=n$.

- ② (x_k^*, y_k^*, z_k^*) an arbitrary point in k th box.
 k th box: side lengths $\Delta x_k, \Delta y_k, \Delta z_k$.
 volume $= \Delta V_k = \Delta x_k \Delta y_k \Delta z_k$

③
$$\sum_{k=1}^n f(x_k^*, y_k^*, z_k^*) \Delta V_k \quad \text{fig 14.38}$$

- ④ Δ : The maximum length of the diagonals of n boxes.

$$\lim_{\Delta \rightarrow 0} \sum_{k=1}^n f(x_k^*, y_k^*, z_k^*) \Delta V_k = \iiint_D f(x, y, z) dV$$

Limit of the Riemann sum.

Application: D is a 3-dim solid and its density varies from point to point according to $f(x, y, z)$.

$$\iiint_D f dV = \text{total mass of } D.$$

Finding limits of integration

fig 14.39

$$\iiint_D f(x, y, z) = \iint_R \left[\int_{G(x, y)}^{H(x, y)} f(x, y, z) dz \right] dA$$

D is bounded by $z = G(x, y)$ from below
and bounded by $z = H(x, y)$ from above

$$\int_{G(x, y)}^{H(x, y)} f(x, y, z) dz$$

partial integral along z
(x, y) fixed.

Projection of D onto the xy plane: region R .

$$\iint_R \left[\int_{G(x, y)}^{H(x, y)} f(x, y, z) dz \right] dA$$

Double integral over R in the xy plane.
(Section 14.2)

fig 14.40

Inner z $G(x, y) \leq z \leq H(x, y)$
middle y $g(x) \leq y \leq h(x)$
outer x $a \leq x \leq b$

$$\int_a^b \int_{g(x)}^{h(x)} \int_{G(x, y)}^{H(x, y)} f(x, y, z) dz dy dx$$

An iterated integral.

Ex. Mass of a rectangular box

$$f(x, y, z) = 2 - z, \quad D: [0, 3] \times [0, 2] \times [0, 1]$$

$$\int_0^3 \int_0^2 \int_0^1 (2 - z) dz dy dx = \int_0^3 \int_0^2 \left(2z - \frac{z^2}{2} \right) \Big|_{z=0}^1 dy dx$$

$$= 3 \times 2 \times \left(2 - \frac{1}{2} \right) = 9.$$

Ex. Volume of the prism D in the 1st octant bounded by $y=4-2x$ (plane) and $z=6$ (plane)

fig 14.42

$$\int_0^2 \int_0^{4-2x} \int_0^6 dz dy dx$$

$$= 6 \int_0^2 (4-2x) dx = 6 (4x - x^2)_0^2 = 24$$

$\int_0^6 dz = 6$ $\int_0^{4-2x} dy = 4-2x$

The order $dz dy dx$ may be changed.

Ex A volume integral

Volume bounded by $y = x^2 + z^2$ and $y = 16 - 3x^2 - z^2$

fig 14.44

inner $\int_{x^2+z^2}^{16-3x^2-z^2} dy$

Region R in xz plane:

$$x^2 + z^2 = 16 - 3x^2 - z^2 \rightarrow 2x^2 + z^2 = 8$$

elliptic region

$$z = \pm \sqrt{8-2x^2}$$

$$\int_{-2}^2 \int_{-\sqrt{8-2x^2}}^{\sqrt{8-2x^2}} \int_{x^2+z^2}^{16-3x^2-z^2} dy dz dx$$

y : inner, z : middle, x : outer.

$$= \int_{-2}^2 \int_{-\sqrt{8-2x^2}}^{\sqrt{8-2x^2}} (16 - 4x^2 - 2z^2) dz dx$$

$$= \int_{-2}^2 \left(16z - 4x^2z - \frac{2}{3}z^3 \right) \Big|_{z=-\sqrt{8-2x^2}}^{\sqrt{8-2x^2}} dx$$

$$= \frac{16\sqrt{2}}{3} \int_{-2}^2 (4-x^2)^{\frac{3}{2}} dx = 32\sqrt{2} \pi$$

$\rightarrow 6\pi$, from $x = 2 \sin \theta$: substitution.

Changing the order of integration

$$\int_0^{\pi^{\frac{1}{4}}} \int_0^z \int_y^z 12y^2 z^3 \sin x^4 \, dx \, dy \, dz$$

What is the solid region D of integration?

Change the order and evaluate the integral.

fig 14.45

① Inner x ; ② region R in yz plane

$$R = \{(y, z): 0 \leq y \leq z, 0 \leq z \leq \pi^{\frac{1}{4}}\}$$

③ A triangular region bounded by $y=0$, $y=z$, $z=\pi^{\frac{1}{4}}$.
Over this region R , the solid D is bounded by four planes: $x=y$, $x=z$, $y=0$, $z=\pi^{\frac{1}{4}}$.
A tetrahedron

New order

$$\begin{aligned} & \int_0^{\pi^{\frac{1}{4}}} \int_0^z \int_y^z 12y^2 z^3 \sin x^4 \, dy \, dx \, dz \\ & \int_0^x 12y^2 dy = 4x^3 \quad 4x^3 dx = d(x^4) \\ & = \int_0^{\pi^{\frac{1}{4}}} \int_0^z \frac{4x^3}{z^3} \sin x^4 \, dx \, dz \\ & = \int_0^{\pi^{\frac{1}{4}}} \left(-z^3 \cos x^4 \Big|_0^z \right) dz \\ & = \int_0^{\pi^{\frac{1}{4}}} (-z^3 \cos z^4 + z^3) dz \\ & = \left(-\frac{1}{4} \sin z^4 + \frac{1}{4} z^4 \right) \Big|_0^{\pi^{\frac{1}{4}}} \\ & = \frac{1}{4} (\pi^{\frac{1}{4}})^4 = \frac{\pi}{4} \end{aligned}$$

Average value

$$\bar{f} = \frac{\iiint_D f(x, y, z) dv}{\iiint_D dv}$$

average of f over D .

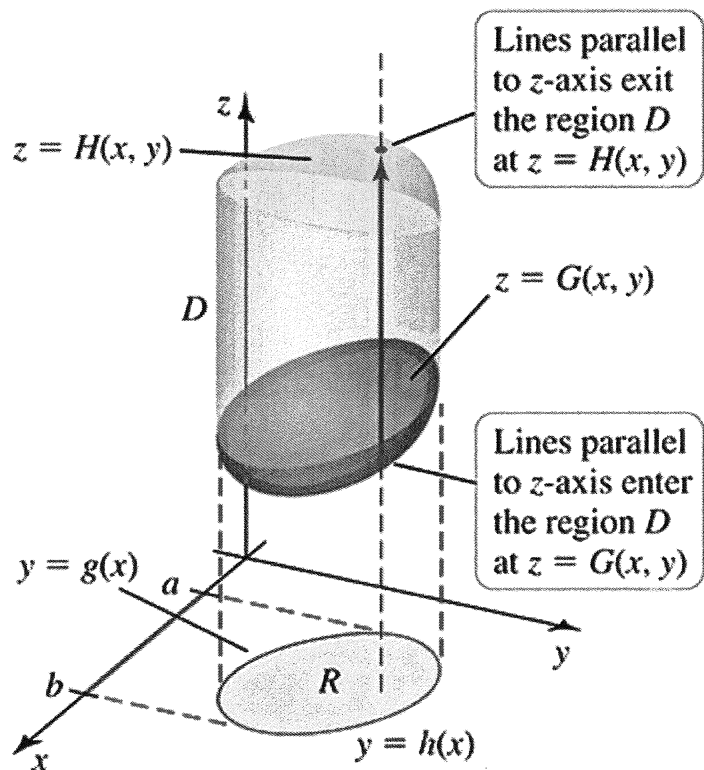
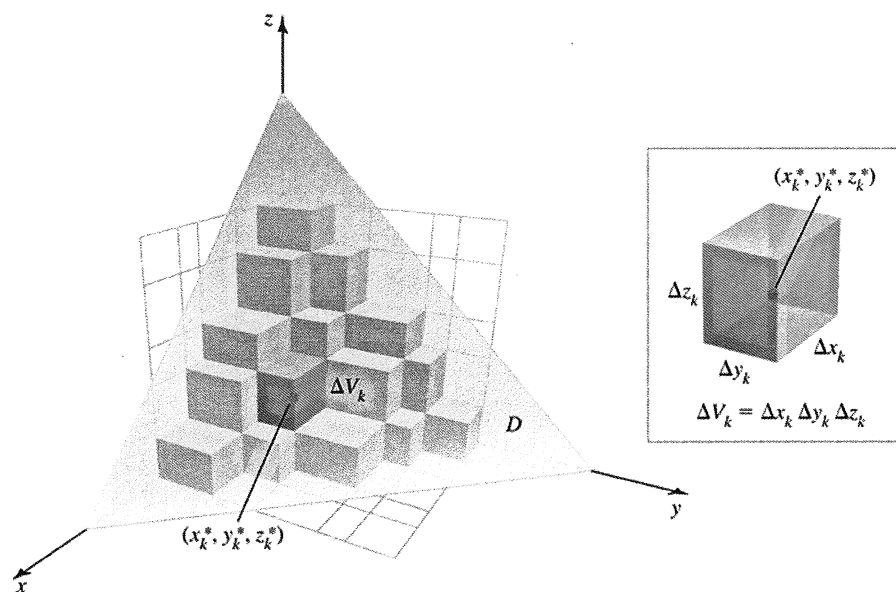
$$\iiint_D dv = \text{Volume of } D$$

Ex. $T(x, y, z) = 250xy \sin \pi z$
 $D = [0, 2] \times [0, 2] \times [0, 1]$

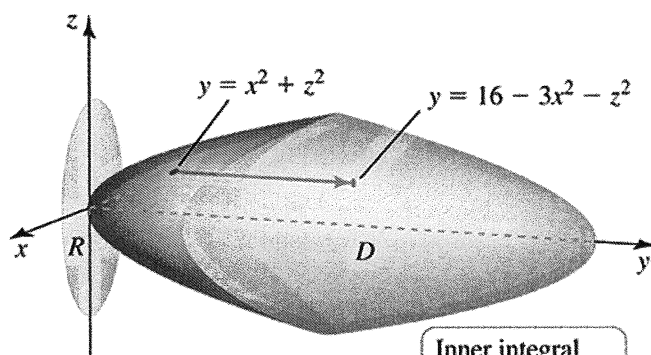
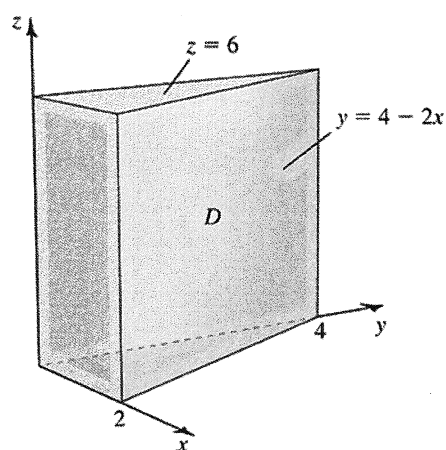
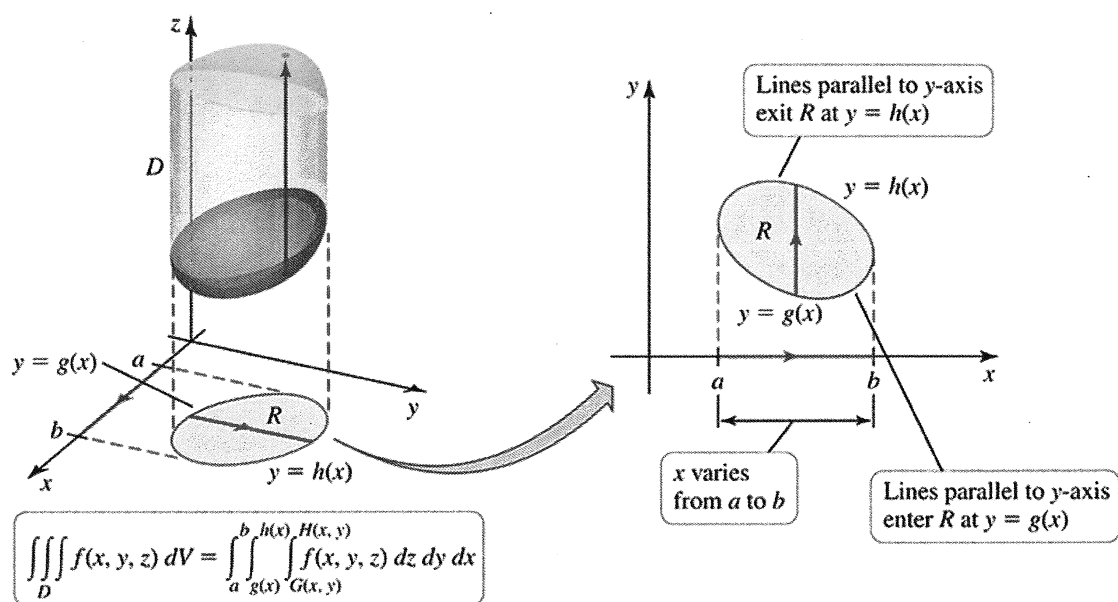
$$\begin{aligned} \iiint_D T dv &= \int_0^2 \int_0^2 \int_0^1 250xy \sin \pi z \, dz \, dy \, dx \\ &= \int_0^2 \int_0^2 250xy \left(-\frac{1}{\pi} \cos \pi z \right) \Big|_0^1 dy \, dx \\ &= \frac{500}{\pi} \int_0^2 \int_0^2 xy \, dy \, dx \\ &= \frac{500}{\pi} \int_0^2 2x \, dx = \frac{2000}{\pi} \end{aligned}$$

$$\text{Volume} = 4$$

$$\bar{T} = \frac{\frac{2000}{\pi}}{4} = \frac{500}{\pi}$$

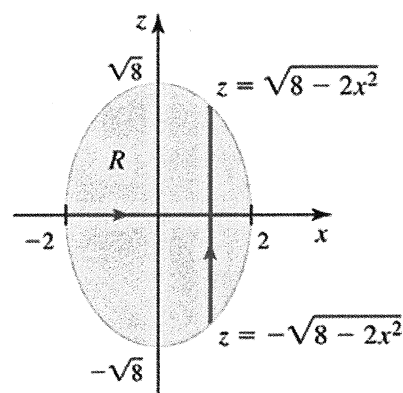


$$\iiint_D f(x, y, z) dV = \iint_R \left[\int_{G(x, y)}^{H(x, y)} f(x, y, z) dz \right] dA$$



Inner integral with respect to y

$$\iint_R \left[\int_{x^2+z^2}^{16-3x^2-z^2} dy \right] dA$$



$$\int_{-2}^2 \int_{-\sqrt{8-2x^2}}^{\sqrt{8-2x^2}} \int_{x^2+z^2}^{16-3x^2-z^2} dy dz dx$$

