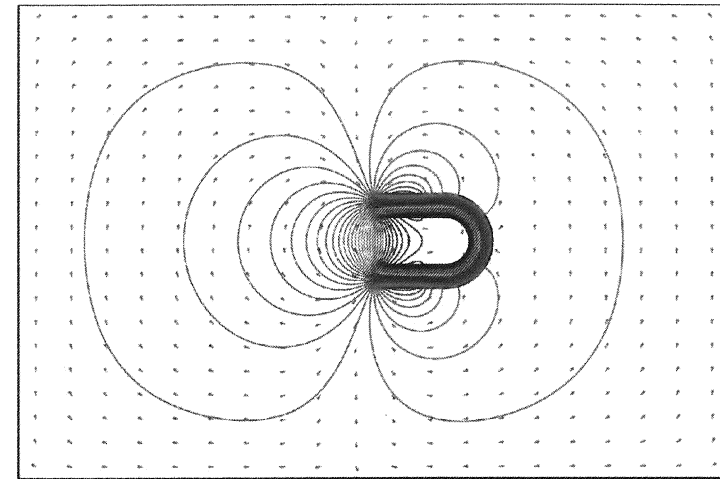
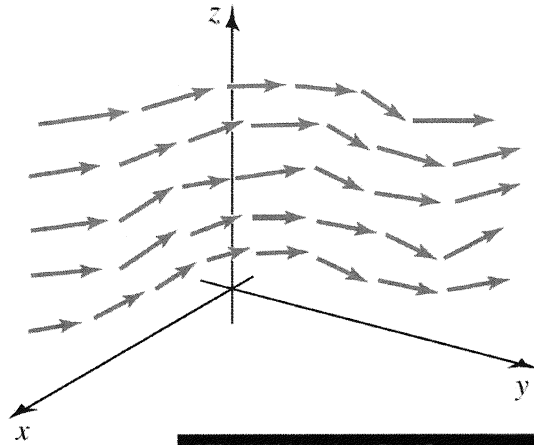


15.1

Vector Fields

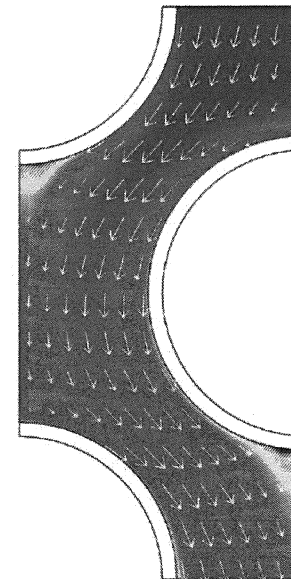
A velocity vector field models the motion of air particles in a breeze at a single moment in time. Individual vectors indicate direction of motion, and their lengths indicate speed.



Magnetic force fields



The wind patterns in a hurricane



The circulation of water in a heat exchanger

Vector Fields in Two Dimensions

DEFINITION Vector Fields in Two Dimensions

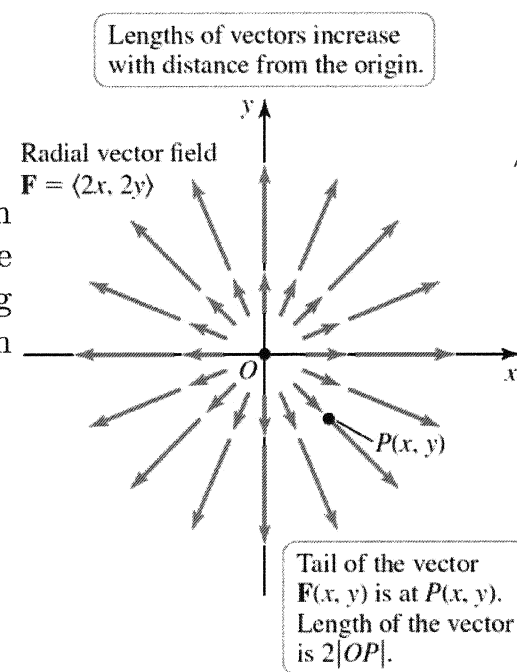
Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point in R a vector $\langle f(x, y), g(x, y) \rangle$. The vector field is written as

$$\mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle \quad \text{or}$$

$$\mathbf{F}(x, y) = f(x, y) \mathbf{i} + g(x, y) \mathbf{j}.$$

A vector field $\mathbf{F} = \langle f, g \rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R , respectively.

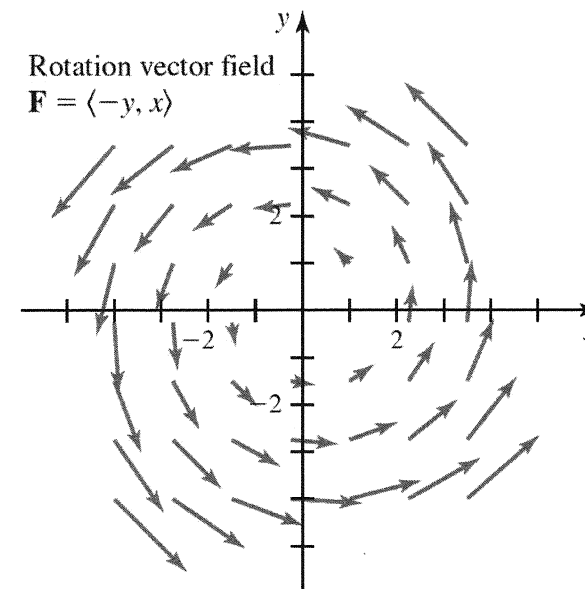
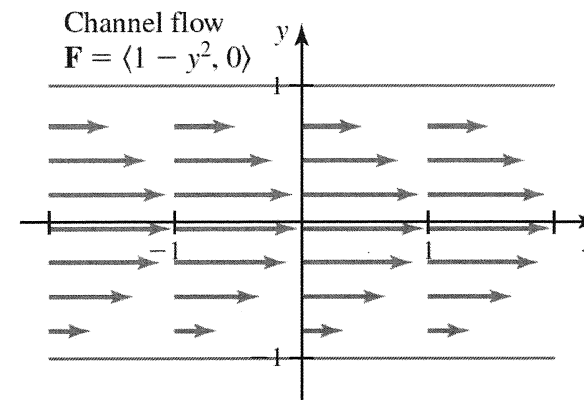
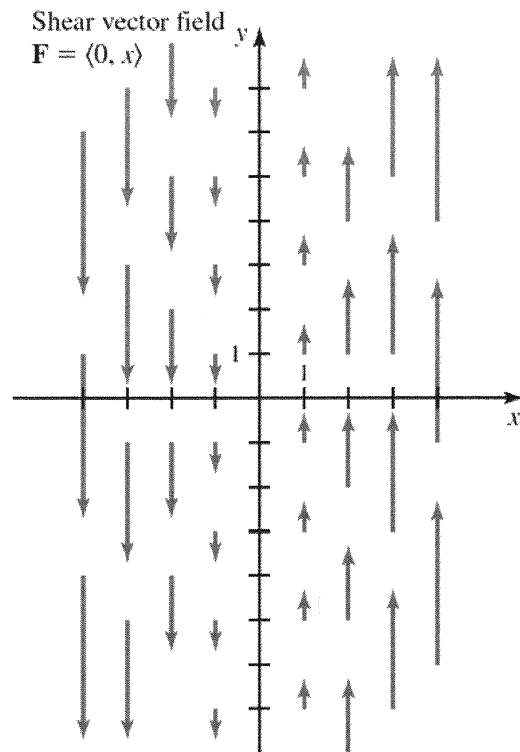
If $\mathbf{F}$ represents the velocity of a fluid moving in two dimensions, the graph of the vector field gives a vivid image of how a small object moves through the fluid. For this reason, vector fields are sometimes called *flows*. When sketching vector fields, it is often useful to draw continuous curves that are aligned with the vector field. Such curves are called **streamlines** or **flow curves**.

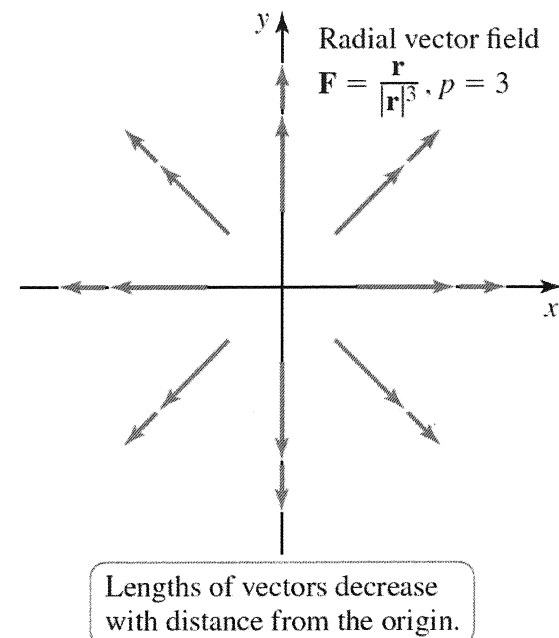
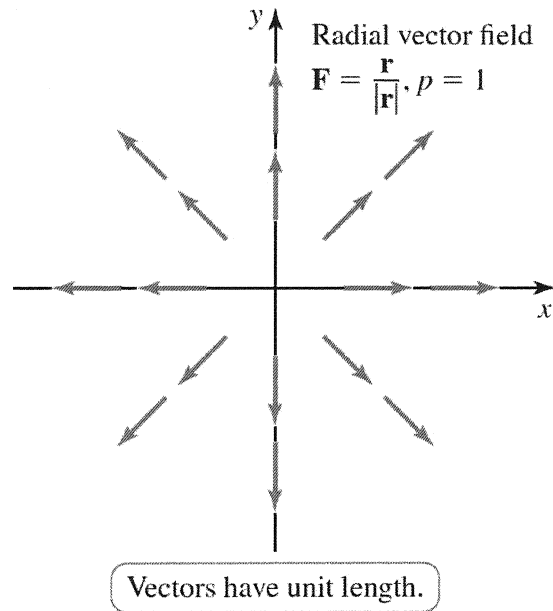


How to represent $\vec{F}(x, y)$ at $P(x, y)$?

EXAMPLE 1 Vector fields Sketch representative vectors of the following vector fields.

- a. $\mathbf{F}(x, y) = \langle 0, x \rangle = x \mathbf{j}$ (a shear field)
- b. $\mathbf{F}(x, y) = \langle 1 - y^2, 0 \rangle = (1 - y^2) \mathbf{i}$, for $|y| \leq 1$ (channel flow)
- c. $\mathbf{F}(x, y) = \langle -y, x \rangle = -y \mathbf{i} + x \mathbf{j}$ (a rotation field)





DEFINITION Radial Vector Fields in $\mathbb{R}^2$

Let $\mathbf{r} = \langle x, y \rangle$. A vector field of the form $\mathbf{F} = f(x, y) \mathbf{r}$, where f is a scalar-valued function, is a **radial vector field**. Of specific interest are the radial vector fields

$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p},$$

where p is a real number. At every point (except the origin), the vectors of this field are directed outward from the origin with a magnitude of $|\mathbf{F}| = \frac{1}{|\mathbf{r}|^{p-1}}$.

EXAMPLE 2 Normal and tangent vectors Let C be the circle $x^2 + y^2 = a^2$, where $a > 0$.

a. Show that at each point of C , the radial vector field $\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$ is orthogonal to the line tangent to C at that point.

b. Show that at each point of C , the rotation vector field $\mathbf{G}(x, y) = \frac{\langle -y, x \rangle}{\sqrt{x^2 + y^2}}$ is parallel to the line tangent to C at that point.

$$\left. \begin{array}{l} \vec{F} \parallel \nabla g \\ \nabla g \perp \vec{T} \end{array} \right\} \vec{F} \perp \vec{T}$$

$$\left. \begin{array}{l} \vec{G} \perp \nabla g \\ \nabla g \perp \vec{T} \end{array} \right\} \vec{G} \parallel \vec{T}$$

$$x^2 + y^2 = a^2 : C$$

C is a level curve ✓

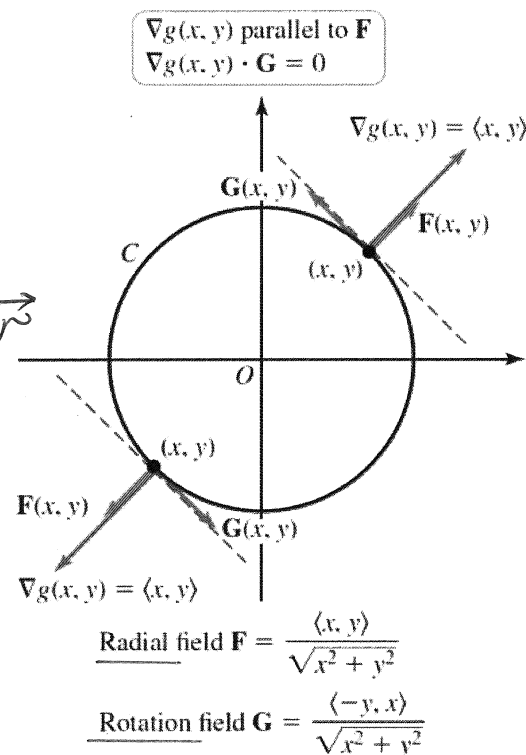
$$\text{of } g(x, y) = x^2 + y^2$$

$$\nabla g(x, y) = \langle 2x, 2y \rangle = 2\vec{r}$$

∇g is everywhere parallel to $\vec{F}(x, y)$.

a. $\vec{F}$ is parallel to ∇g , and ∇g is orthogonal to the tangent vector. So, $\vec{F} \perp$ tangent vector. ✓

b. $\vec{G} \cdot \nabla g = 0$. $\vec{G} \perp \nabla g$
 $\nabla g \perp$ tangent vec. $\} \rightarrow \vec{G} \parallel$ tangent vec.



Vector Fields in Three Dimensions

DEFINITION Vector Fields and Radial Vector Fields in $\mathbb{R}^3$

Let f , g , and h be defined on a region D of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point in D a vector $\langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$. The vector field is written as

$$\begin{aligned}\mathbf{F}(x, y, z) &= \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle \quad \text{or} \\ \mathbf{F}(x, y, z) &= f(x, y, z) \mathbf{i} + g(x, y, z) \mathbf{j} + h(x, y, z) \mathbf{k}.\end{aligned}$$

A vector field $\mathbf{F} = \langle f, g, h \rangle$ is continuous or differentiable on a region D of $\mathbb{R}^3$ if f , g , and h are continuous or differentiable on D , respectively. Of particular importance are the **radial vector fields**

$$\mathbf{F}(x, y, z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z \rangle}{|\mathbf{r}|^p},$$

where p is a real number.

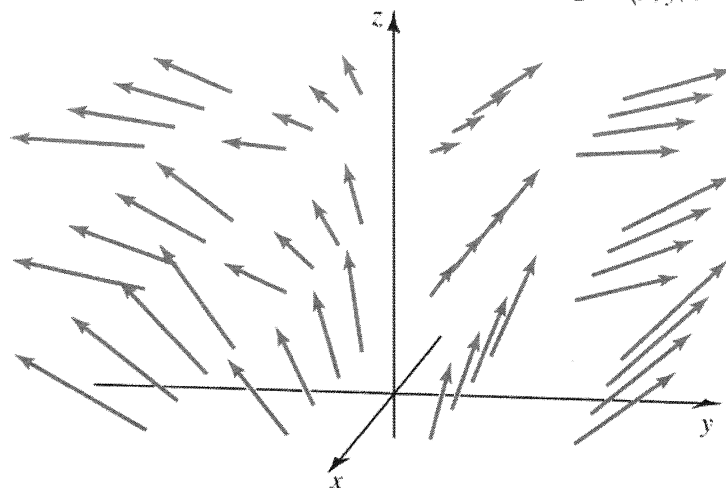
EXAMPLE 3 Vector fields in $\mathbb{R}^3$ Sketch and discuss the following vector fields.

a. $\mathbf{F}(x, y, z) = \langle x, y, e^{-z} \rangle$, for $z \geq 0$

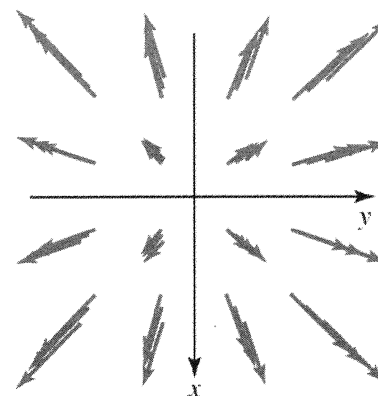
b. $\mathbf{F}(x, y, z) = \langle 0, 0, 1 - x^2 - y^2 \rangle$, for $x^2 + y^2 \leq 1$

$\mathbf{F} = \langle x, y, e^{-z} \rangle$, for $z \geq 0$

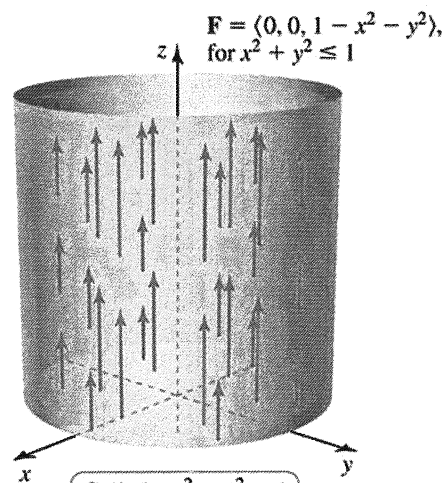
*from the plane of $z=0$ to the planes
of $z=z_0 > 0$.
 $z_0 \rightarrow \infty$, $e^{-z_0} \rightarrow 0$*



View from the side



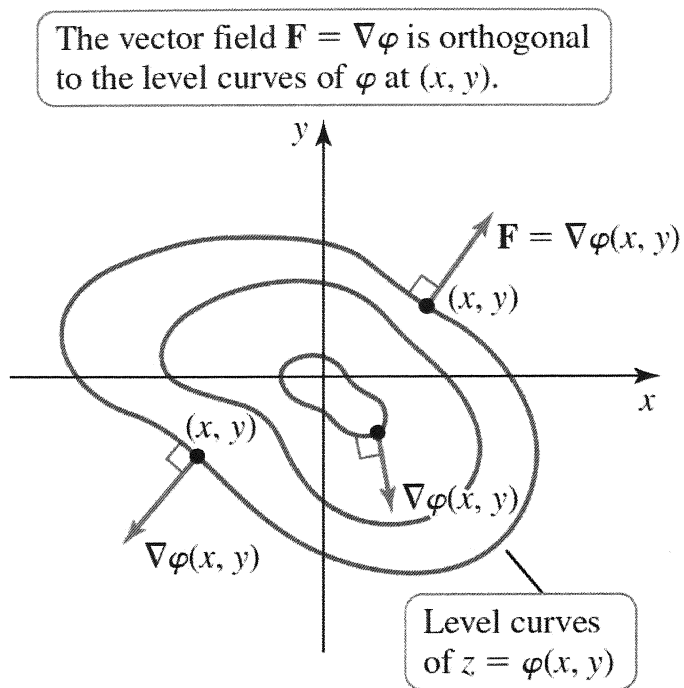
View from above



Cylinder $x^2 + y^2 = 1$

$$h(x, y, z) = 1 - x^2 - y^2 \\ = 1 - r^2$$

flow in a cylindrical tube.



A potential function plays the role of an antiderivative of a vector field: Derivatives of the potential function produce the vector field. If φ is a potential function for a gradient field, then $\varphi + C$ is also a potential function for that gradient field, for any constant C .

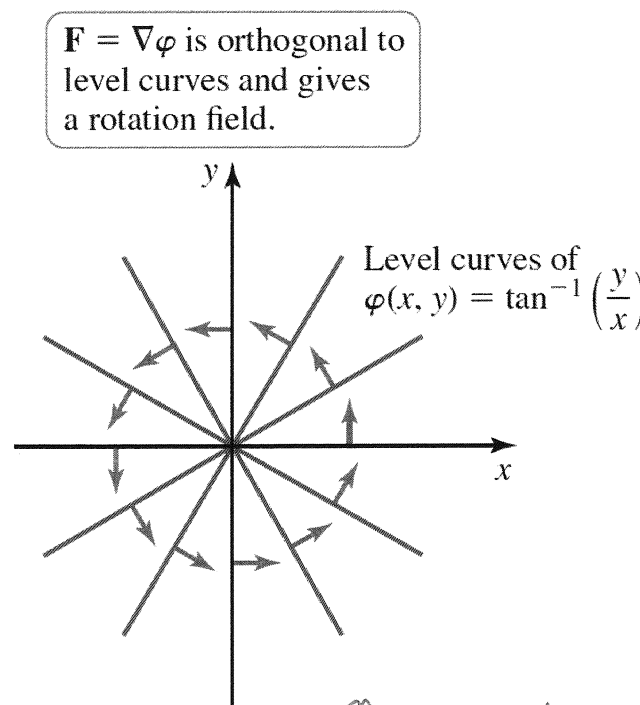
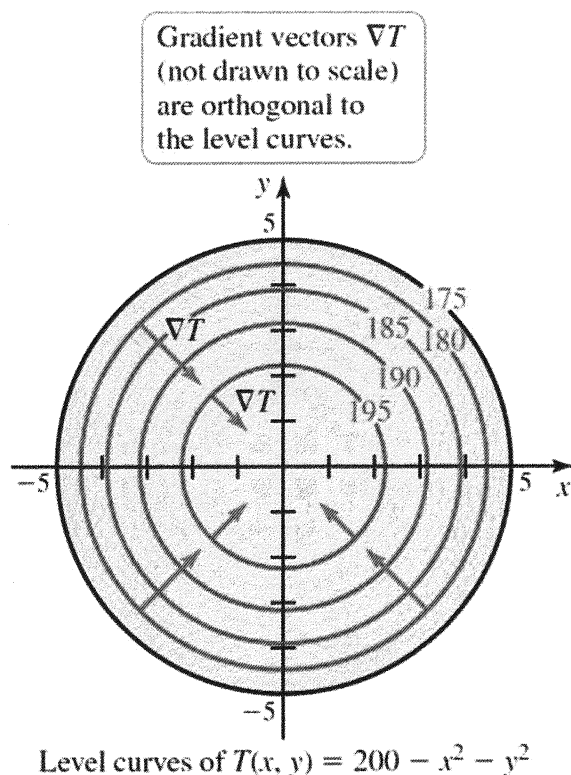
DEFINITION Gradient Fields and Potential Functions

Let $z = \varphi(x, y)$ and $w = \varphi(x, y, z)$ be differentiable functions on regions of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The vector field $\mathbf{F} = \nabla\varphi$ is a **gradient field**, and the function φ is a **potential function** for $\mathbf{F}$.

EXAMPLE 4 Gradient fields

- a. Sketch and interpret the gradient field associated with the temperature function $T = 200 - x^2 - y^2$ on the circular plate $R = \{(x, y) : x^2 + y^2 \leq 25\}$.
- b. Sketch and interpret the gradient field associated with the velocity potential $\varphi = \tan^{-1}(y/x)$.

$$\begin{aligned}\vec{F} &= \nabla T = \langle -2x, -2y \rangle \\ &= -2\vec{r}\end{aligned}$$



$$\begin{aligned}\varphi_x &= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(-\frac{y}{x^2}\right) \\ &= \frac{-y}{x^2 + y^2},\end{aligned}$$

$$\begin{aligned}\varphi_y &= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} \\ &= \frac{x}{x^2 + y^2}\end{aligned}$$

$$\nabla \varphi = \frac{\langle -y, x \rangle}{x^2 + y^2}$$

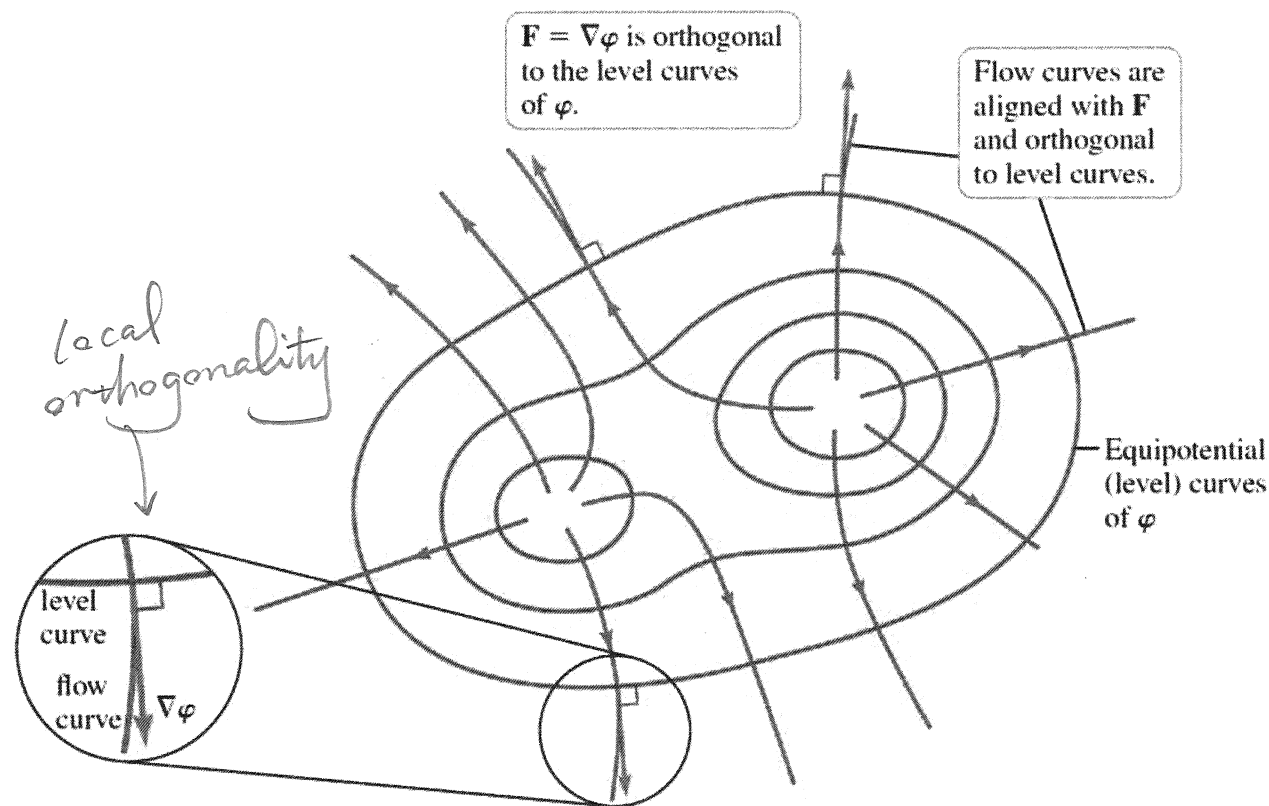
$$\varphi = \arctan u$$

$$u = \tan \varphi$$

$$\frac{du}{d\varphi} = \sec^2 \varphi = 1 + \tan^2 \varphi = 1 + u^2$$

$$\frac{d\varphi}{du} = \frac{1}{1 + u^2} = \frac{1}{1 + \left(\frac{y}{x}\right)^2}$$

Equipotential Curves and Surfaces The preceding example illustrates a beautiful geometric connection between a gradient field and its associated potential function. Let φ be a potential function for the vector field $\mathbf{F}$ in $\mathbb{R}^2$; that is, $\mathbf{F} = \nabla\varphi$. The level curves of a potential function are called **equipotential curves** (curves on which the potential function is constant).



$\nabla\varphi = \vec{F}$
 is orthogonal
 to the level curves
 of φ : equi-
 potential
 curves.

EXAMPLE 5 Equipotential curves The equipotential curves for the potential function $\varphi(x, y) = (x^2 - y^2)/2$ are shown in Figure 15.15.

- Find the gradient field associated with φ and verify that the gradient field is orthogonal to the equipotential curve at $(2, 1)$.
- Verify that the vector field $\mathbf{F} = \nabla\varphi$ is orthogonal to the equipotential curves at all points (x, y) .

$$\nabla\varphi = \langle x, -y \rangle$$

Slope at any point
on a level curve of $\varphi(x, y)$:

$$\varphi(x, y) = \text{Const.}$$

$$\varphi_x + \varphi_y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y} = -\frac{x}{(-y)} = \frac{x}{y}$$

$$\vec{T} \sim \left\langle 1, \frac{x}{y} \right\rangle \sim \langle y, x \rangle$$

$$\vec{T} \cdot \nabla\varphi \sim \langle y, x \rangle \cdot \langle x, -y \rangle = 0$$

