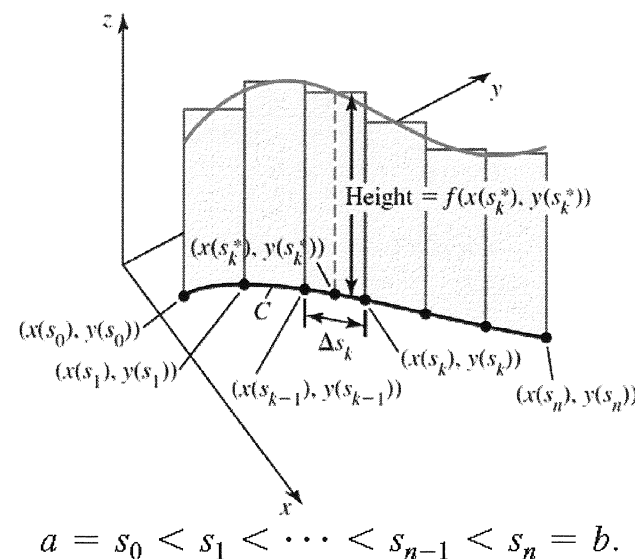
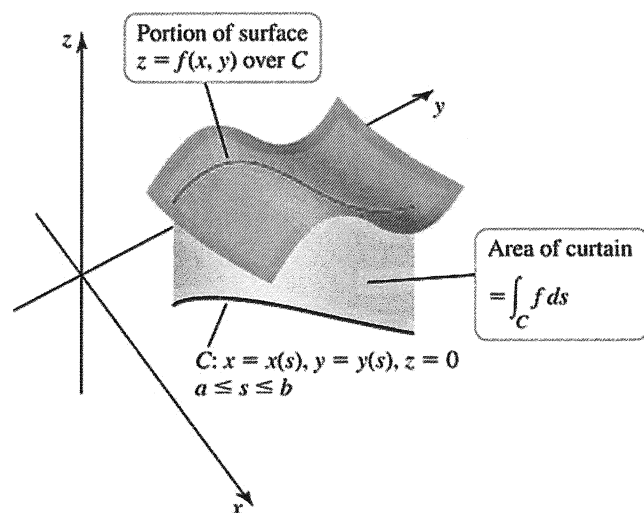


15.2

Line Integrals

Scalar Line Integrals in the Plane



C is a smooth curve of finite length on xy -plane, parametrized in terms of arc length as $\mathbf{r}(s) = \langle x(s), y(s) \rangle$, for $a \leq s \leq b$.

$$\text{area} \approx \sum_{k=1}^n f(x(s_k^*), y(s_k^*)) \Delta s_k.$$

DEFINITION Scalar Line Integral in the Plane, Arc Length Parameter

Suppose the scalar-valued function f is defined on the smooth curve $C: \mathbf{r}(s) = \langle x(s), y(s) \rangle$, parameterized by the arc length s . The **line integral of f over C** is

$$\int_C f(x(s), y(s)) ds = \lim_{\Delta \rightarrow 0} \sum_{k=1}^n f(x(s_k^*), y(s_k^*)) \Delta s_k,$$

provided this limit exists over all partitions of C . When the limit exists, f is said to be **integrable** on C .

EXAMPLE 1 Average temperature on a circle The temperature of the circular plate $R = \{(x, y): x^2 + y^2 \leq 1\}$ is $T(x, y) = 100(x^2 + 2y^2)$. Find the average temperature along the edge of the plate.

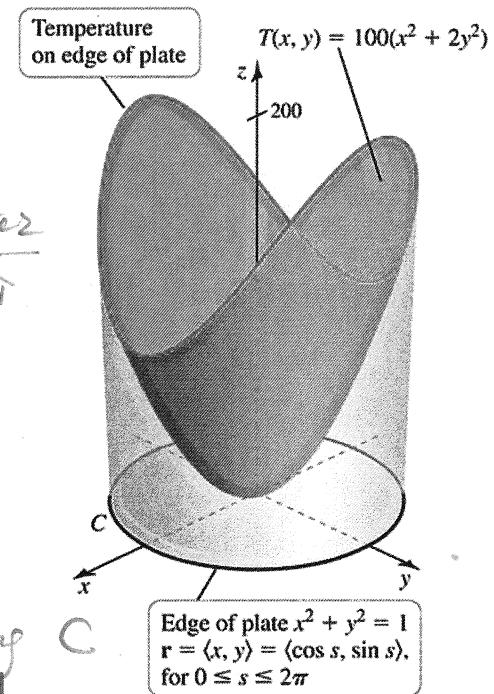
$$\begin{aligned}\vec{r} &= \langle x, y \rangle \\ &= \langle \cos s, \sin s \rangle \\ 0 &\leq s \leq 2\pi \\ s &\text{ is arc length parameter} \\ 0 &\rightarrow 2\pi, \text{ perimeter} = 2\pi\end{aligned}$$

$$\begin{aligned}\int_C T(x, y) ds &= \int_0^{2\pi} 100 (\cos^2 s + 2 \sin^2 s) ds \\ &= 100 (\pi + 2\pi) = 300\pi\end{aligned}$$

$$\frac{300\pi}{2\pi} = 150 : \text{average along } C$$

When we compute the average value by an ordinary integral, we divide by the length of the interval of integration. Analogously, when we compute the average value by a line integral, we divide by the length of the curve L :

$$\bar{f} = \frac{1}{L} \int_C f ds.$$



THEOREM 15.1 Evaluating Scalar Line Integrals in $\mathbb{R}^2$

Let f be continuous on a region containing a smooth curve $C: \mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. Then

$$\begin{aligned}\int_C f ds &= \int_a^b f(x(t), y(t)) |\mathbf{r}'(t)| dt \\ &= \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt.\end{aligned}$$

$$\begin{aligned}ds &= |\vec{r}'(t)| dt \\ \text{from } ds &= \sqrt{(dx)^2 + (dy)^2}\end{aligned}$$

PROCEDURE Evaluating the Line Integral $\int_C f ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$.
2. Compute $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.
3. Make substitutions for x and y in the integrand and evaluate an ordinary integral:

$$\int_C f ds = \int_a^b f(x(t), y(t)) |\mathbf{r}'(t)| dt.$$

EXAMPLE 2 Average temperature on a circle The temperature of the circular plate $R = \{(x, y): x^2 + y^2 \leq 1\}$ is $T(x, y) = 100(x^2 + 2y^2)$ as in Example 1. Confirm the average temperature computed in Example 1 when the circle has the parametric description

$$C = \{(x, y): x = \cos t^2, y = \sin t^2, 0 \leq t \leq \sqrt{2\pi}\}.$$

$$\begin{aligned} x' &= -2t \sin t^2 \\ y' &= 2t \cos t^2 \\ \sqrt{(x')^2 + (y')^2} \, dt &= \underline{\underline{2t \, dt}} \end{aligned}$$

$$\begin{aligned} \int_C T(x, y) \, ds &= \int_0^{\sqrt{2\pi}} 100 (\cos^2 t^2 + 2 \sin^2 t^2) 2t \, dt \\ &= 100 \int_0^{2\pi} (\cos^2 u + 2 \sin^2 u) \, du \quad u = t^2 \\ &= 100 (\pi + 2\pi) = 300\pi \quad \text{as before.} \end{aligned}$$

Line Integrals in $\mathbb{R}^3$

THEOREM 15.2 Evaluating Scalar Line Integrals in $\mathbb{R}^3$

Let f be continuous on a region containing a smooth curve

$C: \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \leq t \leq b$. Then

$$\begin{aligned} \int_C f \, ds &= \int_a^b f(x(t), y(t), z(t)) |\mathbf{r}'(t)| \, dt \\ &= \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt. \end{aligned}$$

EXAMPLE 3 Line integrals in $\mathbb{R}^3$ Evaluate $\int_C (xy + 2z) ds$ on the following line segments.

a. The line segment from $P(1, 0, 0)$ to $Q(0, 1, 1)$

b. The line segment from $Q(0, 1, 1)$ to $P(1, 0, 0)$

a. $\vec{r}(t) = \langle 1, 0, 0 \rangle + t \langle -1, 1, 1 \rangle$

$$|\vec{r}'(t)| = |\langle -1, 1, 1 \rangle| = \sqrt{3}$$

$$\int_C (xy + 2z) ds = \int_0^1 \left[\underbrace{\frac{(1-t)t}{x}}_x + \underbrace{\frac{2t}{2z}}_{2z} \right] \underbrace{\sqrt{3}}_{|\vec{r}'(t)|} dt = \int_0^1 \sqrt{3} (3t - t^2) dt = \sqrt{3} \left(\frac{3}{2} - \frac{1}{3} \right) = \frac{7\sqrt{3}}{6}$$

Similar to a. in every step.

EXAMPLE 4 Flight of an eagle An eagle soars on the ascending spiral path

$$C: \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = \left\langle 2400 \cos \frac{t}{2}, 2400 \sin \frac{t}{2}, 500t \right\rangle,$$

where x , y , and z are measured in feet and t is measured in minutes. How far does the eagle fly over the time interval $0 \leq t \leq 10$?

$$\begin{aligned} \int_C ds &= \int_0^{10} |\vec{r}'(t)| dt \\ &= \int_0^{10} \sqrt{(-1200 \sin \frac{t}{2})^2 + (1200 \cos \frac{t}{2})^2 + (500)^2} dt \quad \nearrow \text{Const.} \\ &= \sqrt{(1200)^2 + (500)^2} \times 10 = \sqrt{169} \times 1000 = \underline{\underline{13,000 \text{ ft.}}} \end{aligned}$$

Line Integrals of Vector Fields

DEFINITION Line Integral of a Vector Field

Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\int_C \mathbf{F} \cdot \mathbf{T} ds$.

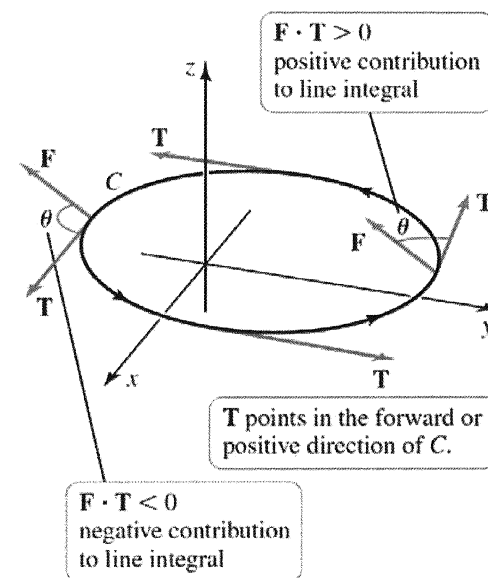
Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f, g, h \rangle$ and C has a parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \leq t \leq b$:

$$\begin{aligned} \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) dt &= \int_a^b (fx'(t) + gy'(t) + hz'(t)) dt \\ &= \int f dx + g dy + h dz \\ &= \underbrace{\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}}_{\mathbf{T}} \underbrace{|\mathbf{r}'(t)| dt}_{ds} = \int_C \mathbf{F} \cdot d\mathbf{r} \end{aligned}$$

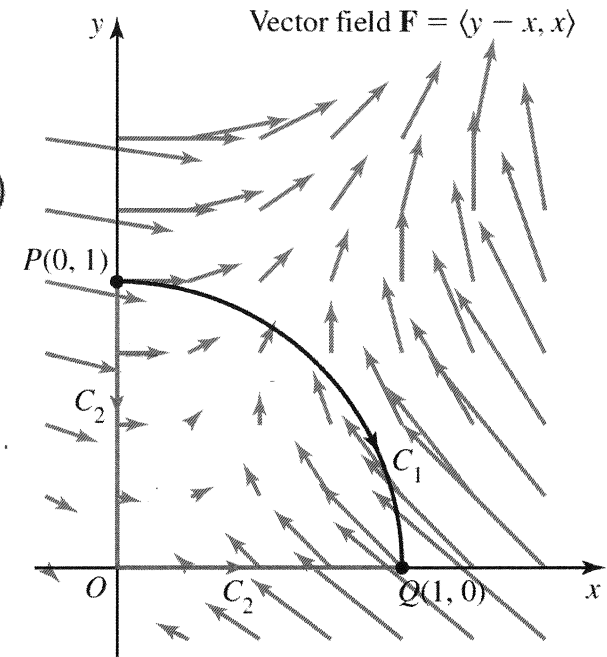
For line integrals in the plane, we let $\mathbf{F} = \langle f, g \rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. Then

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b (fx'(t) + gy'(t)) dt = \int_C f dx + g dy = \int_C \mathbf{F} \cdot d\mathbf{r}.$$



EXAMPLE 5 Different paths Evaluate $\int_C \mathbf{F} \cdot \mathbf{T} ds$ with $\mathbf{F} = \langle y - x, x \rangle$ on the following oriented paths in $\mathbb{R}^2$ (Figure 15.20).

- The quarter circle C_1 from $P(0, 1)$ to $Q(1, 0)$
- The quarter circle $-C_1$ from $Q(1, 0)$ to $P(0, 1)$
- The path C_2 from P to Q via two line segments through $O(0, 0)$



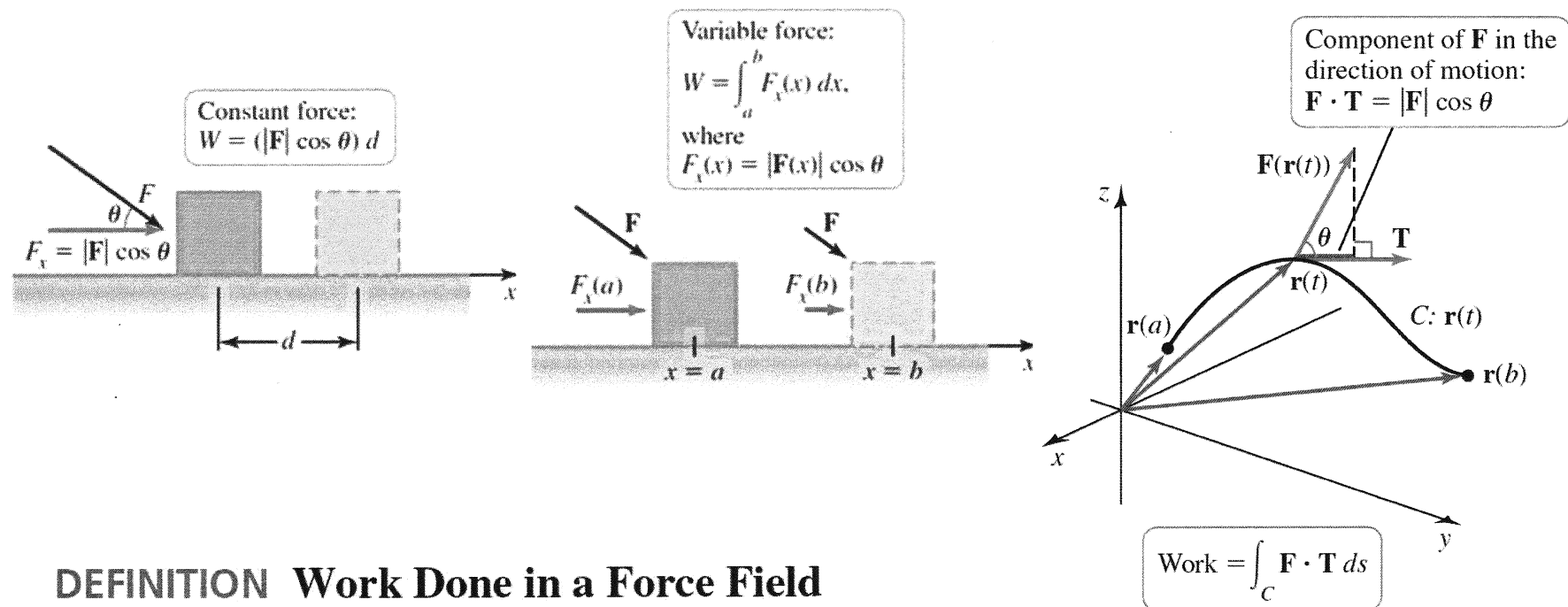
a. $\vec{r}(t) = \langle \sin t, \cos t \rangle$
 parametrization
 $t: 0 \rightarrow \frac{\pi}{2}$
 $\sin t: 0 \rightarrow 1$, $\cos t: 1 \rightarrow 0$

$$\begin{aligned} \int_{C_1} \vec{F} \cdot d\vec{r} &= \int_0^{\frac{\pi}{2}} (\cos t - \sin t) \cos t dt + \sin t (-\sin t) dt \\ &= \int_0^{\frac{\pi}{2}} \left(\cos 2t - \frac{1}{2} \sin 2t \right) dt = \frac{1}{2} \sin 2t \Big|_0^{\frac{\pi}{2}} + \frac{1}{4} \cos 2t \Big|_0^{\frac{\pi}{2}} \\ &= -\frac{1}{2} \\ &= \int_C (y-x) dx + x dy \\ &= (\cos t - \sin t) \cos t dt + \sin t (-\sin t) dt = (\cos 2t - \frac{1}{2} \sin 2t) dt \end{aligned}$$

b. $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $t: 0 \rightarrow \frac{\pi}{2}$, $x = \cos t: 1 \rightarrow 0$, $y = \sin t: 0 \rightarrow 1$
 $(y-x) dx + x dy = (\sin t - \cos t)(-\sin t) dt + \cos t \cos t dt = (\cos 2t + \frac{1}{2} \sin 2t) dt$

$$\int_{-C_1} \vec{F} \cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \left(\cos 2t + \frac{1}{2} \sin 2t \right) dt = \frac{1}{2}$$

21
 $C_1 \Rightarrow -C_1$
 $-\frac{1}{2} \Rightarrow \frac{1}{2}$



DEFINITION Work Done in a Force Field

Let $\mathbf{F}$ be a continuous force field in a region D of $\mathbb{R}^3$ and let $C: \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \leq t \leq b$, be a smooth curve in D with a unit tangent vector $\mathbf{T}$ consistent with the orientation. The work done in moving an object along C in the positive direction is

$$W = \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) dt.$$

EXAMPLE 6 An inverse square force Gravitational and electrical forces between point masses and point charges obey inverse square laws: They act along the line joining the centers and they vary as $1/r^2$, where r is the distance between the centers. The force of attraction (or repulsion) of an inverse square force field is given by the vector field

$$\mathbf{F} = \frac{k\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}, \text{ where } k \text{ is a physical constant. Because } \mathbf{r} = \langle x, y, z \rangle, \text{ this}$$

force may also be written $\mathbf{F} = \frac{k\mathbf{r}}{|\mathbf{r}|^3}$. Find the work done in moving an object along the following paths.

a. C_1 is the line segment from $(1, 1, 1)$ to (a, a, a) , where $a > 1$.

b. C_2 is the extension of C_1 produced by letting $a \rightarrow \infty$.

$$\vec{r}(t) = \langle t, t, t \rangle; \quad t: 1 \rightarrow a > 1.$$

Simple parametrization of a line.

$$\begin{aligned} \vec{F} \cdot d\vec{r} &= \frac{k\langle t, t, t \rangle}{(3t^2)^{3/2}} \cdot \langle dt, dt, dt \rangle = \frac{\cancel{3} k t dt}{\cancel{3} \sqrt{3} t^{3/2}} = \frac{k dt}{\sqrt{3} t^{1/2}} \\ \int_C \vec{F} \cdot d\vec{r} &= \int_1^a \frac{k}{\sqrt{3}} \frac{dt}{t^{1/2}} = -\frac{k}{\sqrt{3}} \frac{1}{t^{1/2}} \Big|_1^a = \frac{k}{\sqrt{3}} \left(1 - \frac{1}{a} \right) \end{aligned}$$

$$a \rightarrow \infty. \quad \int_C \vec{F} \cdot d\vec{r} \rightarrow \frac{k}{\sqrt{3}}, \text{ finite.}$$

Circulation and Flux of a Vector Field

DEFINITION Circulation

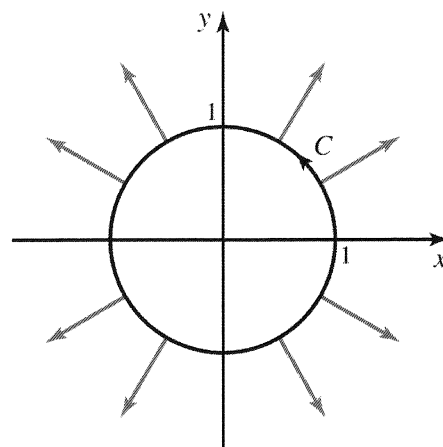
Let $\mathbf{F}$ be a continuous vector field on a region D of $\mathbb{R}^3$ and let C be a closed smooth oriented curve in D . The **circulation** of $\mathbf{F}$ on C is $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.

EXAMPLE 7 Circulation of two-dimensional flows Let C be the unit circle with counterclockwise orientation. Find the circulation on C for the following vector fields.

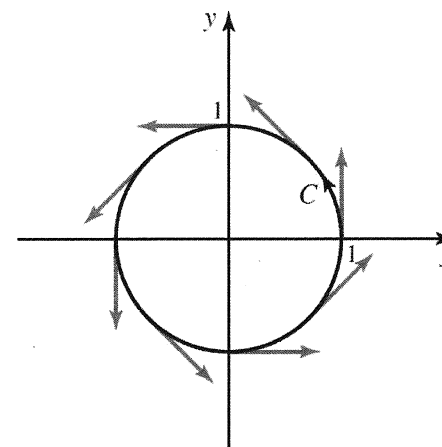
- The radial flow field $\mathbf{F} = \langle x, y \rangle$
- The rotation flow field $\mathbf{F} = \langle -y, x \rangle$

$$\begin{aligned} x &= \cos \theta, \quad y = \sin \theta \\ 0 &\leq \theta \leq 2\pi \\ &\text{for unit circle's} \\ &\text{parametrization} \end{aligned}$$

$$\begin{aligned} \text{a.} \quad & x \, dx + y \, dy \\ &= \cos \theta (-\sin \theta) \, d\theta + \sin \theta \cos \theta \, d\theta \\ &= 0 \\ \int_C \vec{F} \cdot d\vec{r} &= 0 \end{aligned}$$



On the unit circle, $\mathbf{F} = \langle x, y \rangle$ is orthogonal to C and has zero circulation on C .

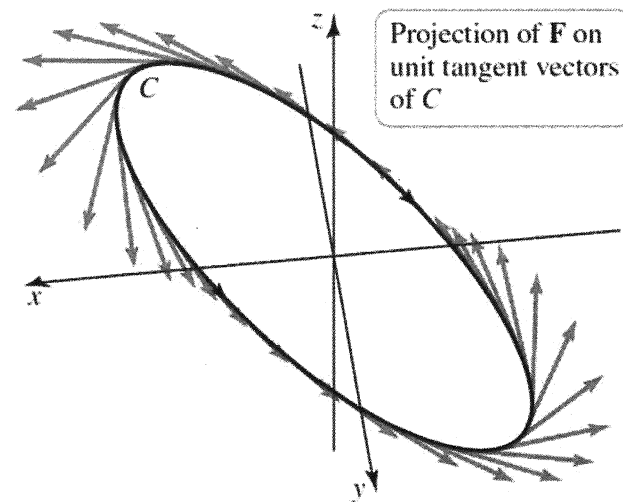
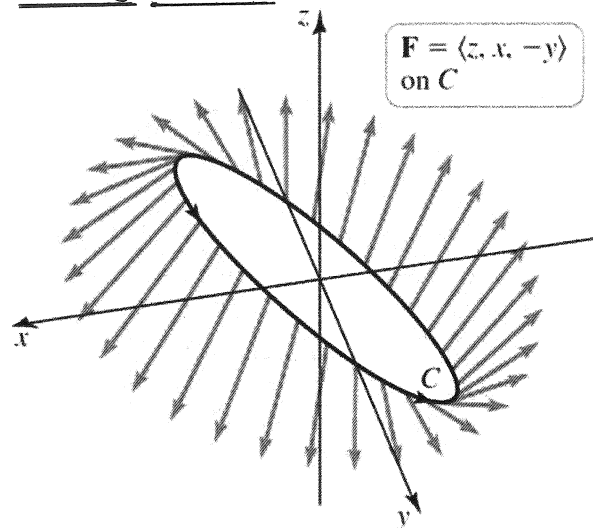


On the unit circle, $\mathbf{F} = \langle -y, x \rangle$ is tangent to C and has positive circulation on C .

$$\begin{aligned} \text{b.} \quad & -y \, dx + x \, dy = -\sin \theta (-\sin \theta) \, d\theta + \cos \theta \cos \theta \, d\theta \\ &= d\theta \\ \int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} d\theta = 2\pi. \end{aligned}$$

EXAMPLE 8 Circulation of a three-dimensional flow Find the circulation of the vector field $\mathbf{F} = \langle z, x, -y \rangle$ on the tilted ellipse $C: \mathbf{r}(t) = \langle \cos t, \sin t, \cos t \rangle$, for $0 \leq t \leq 2\pi$

$$\underline{\underline{z dx + x dy - y dz}}$$



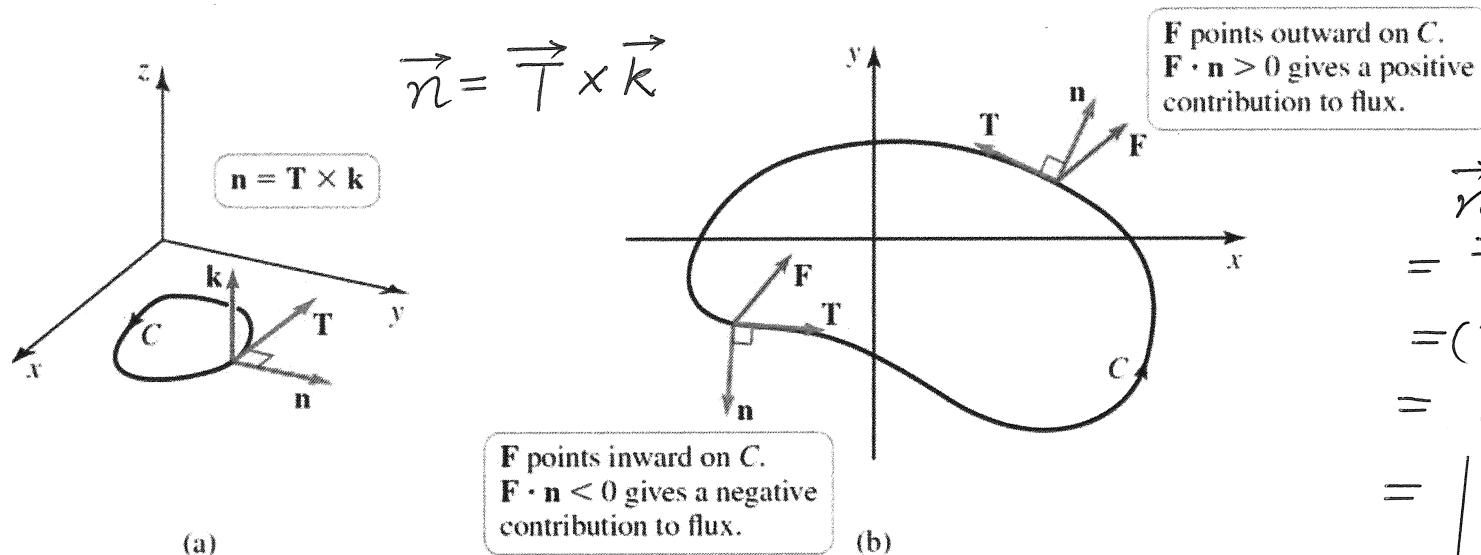
$$\vec{F} \cdot d\vec{r} = z dx + x dy - y dz$$

$$= \cos t (-\sin t) dt + \cos t \cos t dt - \sin t (-\sin t) dt$$

$$= \cos t (-\sin t) dt + \cos t \cos t dt - \sin t (-\sin t) dt$$

$$= \left(-\frac{1}{2} \sin 2t + 1 \right) dt$$

$$\int_0^{2\pi} \left(-\frac{1}{2} \sin 2t + 1 \right) dt = 2\pi = \int_C \vec{F} \cdot d\vec{r}$$



$$\begin{aligned}
 & \vec{n} \, ds \\
 &= \vec{T} \times \vec{k} \, ds \\
 &= (\vec{T} \, ds) \times \vec{k} \\
 &= d\vec{r} \times \vec{k} \\
 &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & dy & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
 &= \langle dy, -dx, 0 \rangle
 \end{aligned}$$

DEFINITION Flux

Let $\mathbf{F} = \langle f, g \rangle$ be a continuous vector field on a region R of $\mathbb{R}^2$. Let $C: \mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$, be a smooth oriented curve in R that does not intersect itself. The **flux** of the vector field across C is

$$\begin{aligned}
 \int_C \mathbf{F} \cdot \mathbf{n} \, ds &= \int_a^b (f y'(t) - g x'(t)) \, dt, \\
 \vec{F} \cdot \vec{n} \, ds &= f \, dy - g \, dx \\
 &= (f y' - g x') \, dt
 \end{aligned}$$

where $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector and the flux integral gives the **outward flux** across C .

$$x = \cos \theta, y = \sin \theta$$

$$0 \leq \theta \leq 2\pi$$

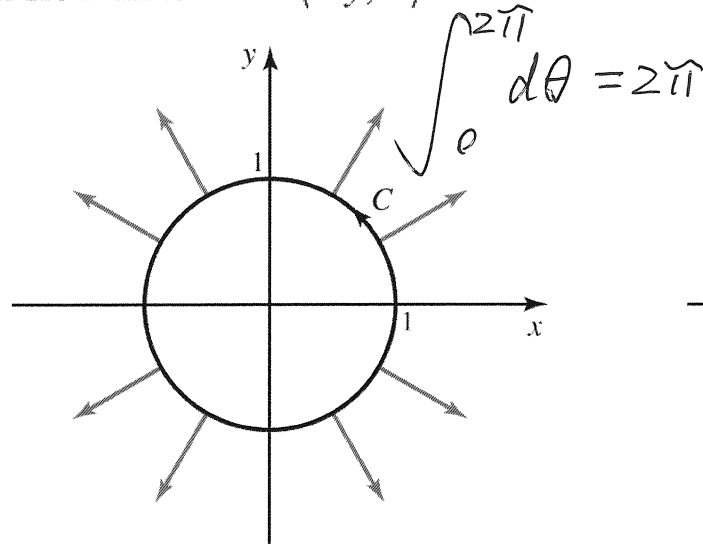
unit circle parametrization

EXAMPLE 9 Flux of two-dimensional flows Find the outward flux across the unit circle with counterclockwise orientation for the following vector fields.

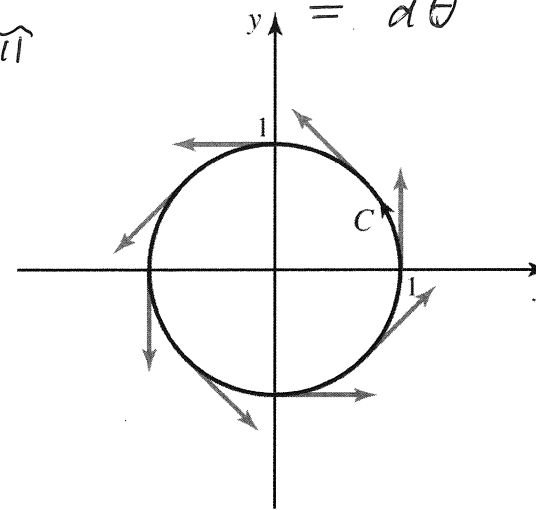
a. The radial vector field $\mathbf{F} = \langle x, y \rangle$

b. The rotation flow field $\mathbf{F} = \langle -y, x \rangle$

a. $x dy - y dx = \cos \theta \cos \theta d\theta - \sin \theta (-\sin \theta) d\theta$
 $= d\theta$



On the unit circle, $\mathbf{F} = \langle x, y \rangle$ is orthogonal to C and has positive outward flux on C .



On the unit circle, $\mathbf{F} = \langle -y, x \rangle$ is tangent to C and has zero outward flux on C .

no flux in b.

b
 $-y dy - x dx = 0$

a. $\int_C \vec{F} \cdot \vec{n} ds = \int_0^{2\pi} d\theta = 2\pi$
 outgoing flux = 2π .