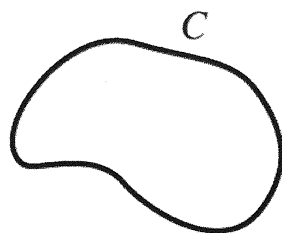


15.3

Conservative Vector Fields

Types of Curves and Regions



Closed, simple



Not closed, simple



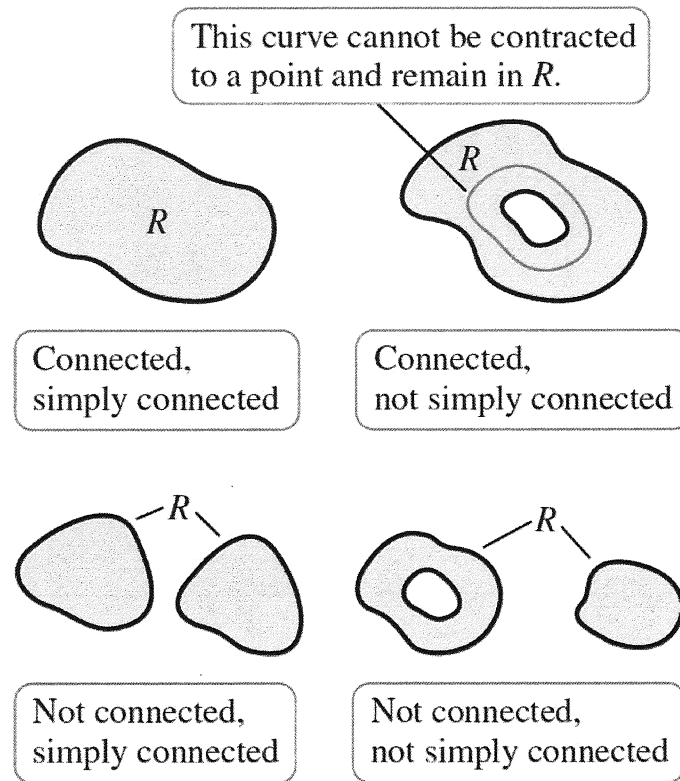
Closed, not simple



Not closed, not simple

DEFINITION Simple and Closed Curves

Suppose a curve C (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \leq t \leq b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all t_1 and t_2 , with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same



DEFINITION Connected and Simply Connected Regions

An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it is possible to connect any two points of R by a continuous curve lying in R . An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R .

Test for Conservative Vector Fields

DEFINITION Conservative Vector Field

A vector field $\mathbf{F}$ is said to be **conservative** on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

potential function φ

THEOREM 15.3 Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f, g, h \rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f , g , and h have continuous first partial derivatives on D .

Then $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

$f_y = g_x$
because of
 $\varphi_{xy} = \varphi_{yx}$

$f_z = h_x$
because of
 $\varphi_{xz} = \varphi_{zx}$, ...

EXAMPLE 1 Testing for conservative fields Determine whether the following vector fields are conservative on $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

a. $\mathbf{F} = \langle e^x \cos y, -e^x \sin y \rangle$

b. $\mathbf{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$

a. $\underline{f_y = -e^x \sin y}, \quad \underline{g_x = -e^x \sin y}, \quad \text{yes, conservative.}$

b. $f_y = g_x = 2x, \quad f_z = h_x = -2z, \quad g_z = h_y = 2, \quad \text{yes,}$
conservative.

Finding Potential Functions

EXAMPLE 2 Finding potential functions Find a potential function for the conservative vector fields in Example 1.

a. $\mathbf{F} = \langle e^x \cos y, -e^x \sin y \rangle$

b. $\mathbf{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$

a. $\varphi_x = e^x \cos y, \quad \varphi = \int \varphi_x dx = e^x \cos y + \underline{c(y)} \rightarrow \text{to be determined.}$
 $\varphi_y = -e^x \sin y + c'(y)$
 $= g = -e^x \sin y, \quad c'(y) = 0, \quad c(y) = \text{Const.}$

$\varphi = e^x \cos y + \text{Const.}$

b. $\varphi_x = 2xy - z^2, \quad \varphi = \int \varphi_x dx = x^2 y - xz^2 + C(y, z)$
 $\varphi_y = x^2 + C_y(y, z) = x^2 + 2z, \quad C_y = 2z, \quad C = \int C_y dy$
 $\varphi = x^2 y - xz^2 + 2yz + d(z), \quad \varphi_z = -2xz + 2y + d' = 2yz + d(z)$
 $\varphi = x^2 y - xz^2 + 2yz + \text{Const.}, \quad = 2y - 2xz, \quad d' = 0, \quad d = \text{Const.}$

PROCEDURE Finding Potential Functions in $\mathbb{R}^3$

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, take the following steps:

1. Integrate $\varphi_x = f$ with respect to x to obtain φ , which includes an arbitrary function $c(y, z)$.
2. Compute φ_y and equate it to g to obtain an expression for $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$, including an arbitrary function $d(z)$.
4. Compute φ_z and equate it to h to get $d(z)$.

Beginning the procedure with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases.

Fundamental Theorem for Line Integrals and Path Independence

THEOREM 15.4 Fundamental Theorem for Line Integrals

Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$). There exists a potential function φ with $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative) if and only if

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

for all points A and B in R and all smooth oriented curves C from A to B .

$$\begin{aligned} \vec{F} \cdot \vec{T} \, ds &= \underline{\underline{\vec{F} \cdot d\vec{r}}} & \vec{T} \, ds &= d\vec{r} \\ &= \langle \varphi_x, \varphi_y, \varphi_z \rangle \cdot \langle dx, dy, dz \rangle \\ &= \langle \varphi_x, \varphi_y, \varphi_z \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle dt \\ &= (\varphi_x x' + \varphi_y y' + \varphi_z z') dt \\ &= \frac{d\varphi}{dt} dt = d\varphi & \varphi &= \varphi(x(t), y(t), z(t)) \\ & & \int_C d\varphi &= \varphi(B) - \varphi(A) \end{aligned}$$

EXAMPLE 3 Verifying path independence Consider the potential function

$\varphi(x, y) = (x^2 - y^2)/2$ and its gradient field $\mathbf{F} = \langle x, -y \rangle$. Let C_1 be the quarter circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $0 \leq t \leq \pi/2$, from $A(1, 0)$ to $B(0, 1)$. Let C_2 be the line $\mathbf{r}(t) = \langle 1 - t, t \rangle$, for $0 \leq t \leq 1$, also from A to B . Evaluate the line integrals of $\mathbf{F}$ on C_1 and C_2 , and show that both are equal to $\varphi(B) - \varphi(A)$.

$$\begin{aligned}
 C_1: & \quad \langle \cos t, -\sin t \rangle \cdot \langle -\sin t, \cos t \rangle dt \\
 & \quad = -2 \sin t \cos t dt \quad \text{for } \vec{F} \cdot d\vec{r} \\
 & \quad \int_0^{\pi/2} -2 \sin t \cos t dt = -\sin^2 t \Big|_0^{\pi/2} = -1 \\
 C_2: & \quad \langle 1-t, -t \rangle \cdot \langle -1, 1 \rangle dt \\
 & \quad = -dt \quad \text{for } \vec{F} \cdot d\vec{r} \quad \int_0^1 (-dt) = -1
 \end{aligned}
 \quad \left. \vphantom{\int_0^1} \right\} \begin{array}{l} \text{equal,} \\ \text{as} \\ \text{expected.} \end{array}$$

EXAMPLE 4 Line integral of a conservative vector field Evaluate

$$\int_C ((2xy - z^2) \mathbf{i} + (x^2 + 2z) \mathbf{j} + (2y - 2xz) \mathbf{k}) \cdot d\mathbf{r},$$

where C is a simple curve from $A(-3, -2, -1)$ to $B(1, 2, 3)$.

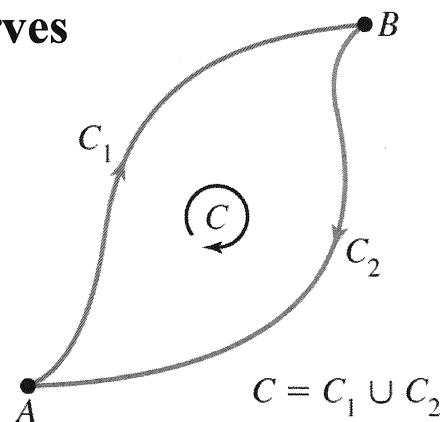
$$\varphi = x^2 y - x z^2 + z y z, \quad \text{derived before.}$$

$$\varphi(A) = -18 + 3 + 4 = -11,$$

$$\varphi(B) = 2 - 9 + 12 = 5$$

$$\int_C \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A) = 16.$$

Line Integrals on Closed Curves



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

THEOREM 15.5 Line Integrals on Closed Curves

Let R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) be an open region. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed smooth oriented curves C in R .

EXAMPLE 5 A closed curve line integral in $\mathbb{R}^3$ Evaluate

$\int_C \nabla(-xy + xz + yz) \cdot d\mathbf{r}$ on the curve $C: \mathbf{r}(t) = \langle \sin t, \cos t, \sin t \rangle$, for $0 \leq t \leq 2\pi$, without using Theorems 15.4 or 15.5.

$$\begin{aligned} \nabla(-xy + xz + yz) \\ = \langle -y + z, -x + z, x + y \rangle \end{aligned}$$

$$\begin{aligned} & \nabla \phi \cdot d\mathbf{r} \\ &= \langle -\cos t + \sin t, 0, \sin t + \cos t \rangle \cdot \langle \cos t, -\sin t, \cos t \rangle dt \\ &= (-\cos^2 t + \sin t \cos t + \sin t \cos t + \cos^2 t) dt \\ &= \sin 2t dt \end{aligned}$$

$\int_0^{2\pi} \sin 2t dt = 0$, verified to be zero.

Summary of the Properties of Conservative Vector Fields

We have established three equivalent properties of conservative vector fields $\mathbf{F}$ defined on an open connected region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$).

- There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ (definition).
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in R and all smooth oriented curves C from A to B (path independence).
- $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple smooth closed oriented curves C in R .

The connections between these properties were established by Theorems 15.4 and 15.5 in the following way:

$$\text{Path-independence} \xLeftrightarrow{\text{Theorem 15.4}} \mathbf{F} \text{ is conservative } (\nabla\varphi = \mathbf{F}) \xLeftrightarrow{\text{Theorem 15.5}} \oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$