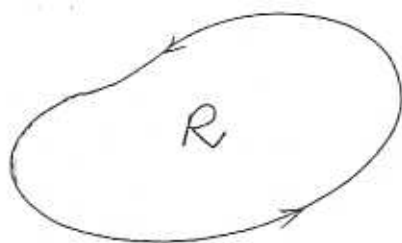


16.4

Green's Theorem

16-4-1

Theorem

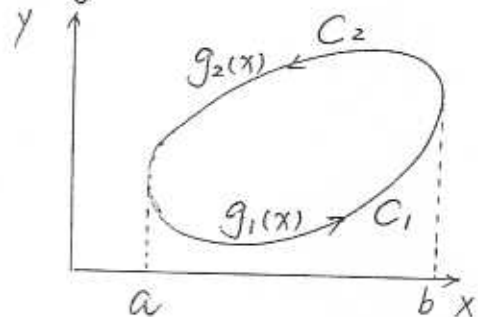


R is a simply connected region (no hole).
The boundary of R is a closed, piecewise smooth curve C oriented counter-clockwise.

$f(x,y)$ and $g(x,y)$ are continuous and have continuous first derivatives on some open set containing R .

$$\int_C f dx + g dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

Proof:



R as a type I region.

$$\begin{aligned} \int_C f dx &= \int_{C_1} f dx + \int_{C_2} f dx \\ &= \int_{C_1} f dx - \int_{-C_2} f dx \end{aligned}$$

$$C_1 : x=t, \quad y=g_1(t), \quad a \leq t \leq b$$

$$-C_2 : x=t, \quad y=g_2(t), \quad a \leq t \leq b$$

$$\begin{aligned} \int_C f dx &= \int_a^b f(t, g_1(t)) x'(t) dt \quad \leftarrow C_1 \\ &\quad - \int_a^b f(t, g_2(t)) x'(t) dt \quad \leftarrow -C_2 \\ &= \int_a^b f(t, g_1(t)) dt - \int_a^b f(t, g_2(t)) dt \quad (x'(t)=1) \end{aligned}$$

$$= - \int_a^b [f(t, g_2(t)) - f(t, g_1(t))] dt$$

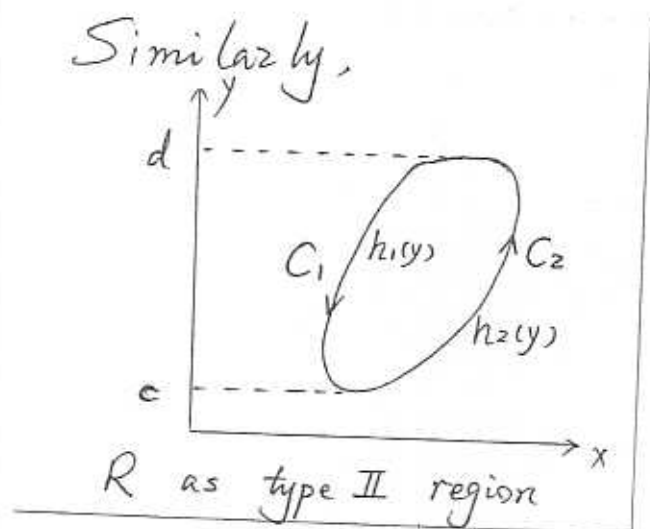
$$= - \int_a^b \left[f(t, y) \right]_{y=g_1(t)}^{y=g_2(t)} dt = - \int_a^b \left[\int_{g_1(t)}^{g_2(t)} \frac{\partial f}{\partial y} dy \right] dt$$

$$= - \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial f}{\partial y} dy dx = - \iint_R \frac{\partial f}{\partial y} dA$$

type I, $\int dy$ done first.

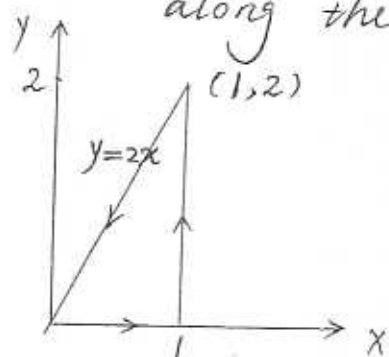
16-4-2

Similarly,



$$\begin{aligned}
 \int_C g \, dy &= \int_{C_1} g \, dy + \int_{C_2} g \, dy \\
 &= \int_{C_2} g \, dy - \int_{-C_1} g \, dy \\
 &= \int_c^d g(h_2(y), y) \, dy - \int_c^d g(h_1(y), y) \, dy \\
 &= \int_c^d [g(h_2(y), y) - g(h_1(y), y)] \, dy \\
 &= \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial g}{\partial x} \, dx \, dy = \iint_R \frac{\partial g}{\partial x} \, dA
 \end{aligned}$$

Ex. Use GT to evaluate $\int_C x^2 y \, dx + x \, dy$ along the triangular path.



piecewise
smooth,
closed curve.

$$\begin{aligned}
 f &= x^2 y, \quad g = x \\
 \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} &= 1 - x^2 \\
 \int_C f \, dx + g \, dy &= \iint_R (1 - x^2) \, dA \\
 &= \int_0^1 \int_0^{2x} (1 - x^2) \, dy \, dx = \int_0^1 2x(1 - x^2) \, dx \\
 &= x^2 \Big|_0^1 - \frac{1}{2} x^4 \Big|_0^1 = \frac{1}{2}
 \end{aligned}$$

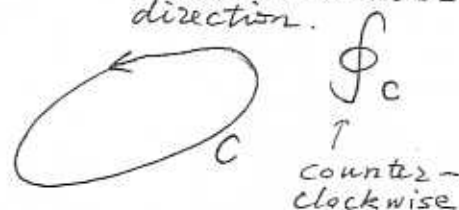
This evaluation by $\iint_R \dots \, dA$ is simpler than evaluating the line integrals along three lines.

Notation $\oint_C f \, dx + g \, dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dA$

$\oint_C$ for line integral along a closed curve with counter-clockwise direction.

counter-clock-wise direction.

$\oint \rightarrow$ clockwise



Ex $\vec{F} = (e^x - y^3) \vec{i} + (\cos y + x^3) \vec{j}$ 16-4-3
 find $\oint_C \vec{F} \cdot d\vec{r}$ around the unit circle $x^2 + y^2 = 1$ in counter clockwise direction.

$$\frac{\partial f}{\partial y} = -3y^2, \quad \frac{\partial g}{\partial x} = 3x^2$$

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = 3 \iint_R (x^2 + y^2) dA \\ &= 3 \int_0^{2\pi} \int_0^1 \underbrace{(r^2)}_{x^2+y^2} \underbrace{r dr d\theta}_{dA} = \frac{3}{4} \int_0^{2\pi} d\theta = \frac{3}{2} \pi \end{aligned}$$

Finding area of a region



$$A = \iint_R dA = \oint_C x dy$$

$$\left(\begin{array}{l} f=0 \\ g(x,y)=x \end{array} \right)$$

$$A = \iint_R dA = - \oint_C y dx$$

$$\left(\begin{array}{l} f=-y \\ g=0 \end{array} \right)$$

$$A = \oint_C x dy = - \oint_C y dx = \frac{1}{2} \oint_C -y dx + x dy$$

Ex. Area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Parametrization $x = a \cos t$, $y = b \sin t$, $0 \leq t \leq 2\pi$.

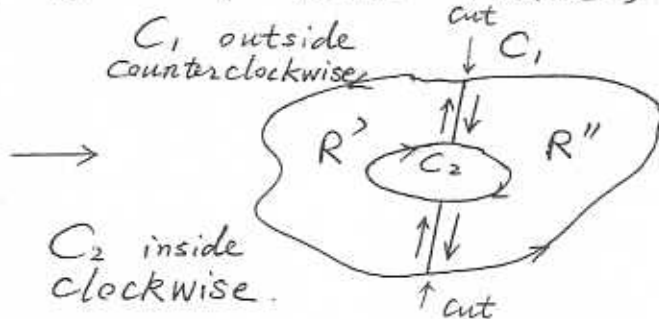
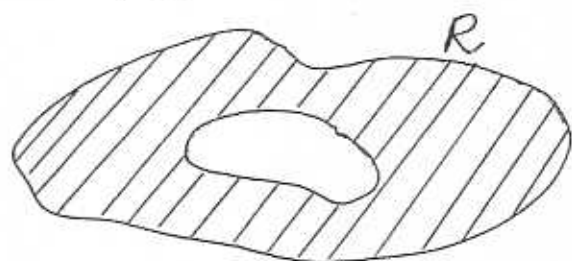
$$A = \frac{1}{2} \oint_C -y dx + x dy$$

$$= \frac{1}{2} \int_0^{2\pi} ((-y)x' + x y') dt$$

$$= \frac{1}{2} \int_0^{2\pi} ((-b \sin t)(-a \sin t) + (a \cos t)(b \cos t)) dt$$

$$= \frac{1}{2} ab \int_0^{2\pi} dt = \pi ab.$$

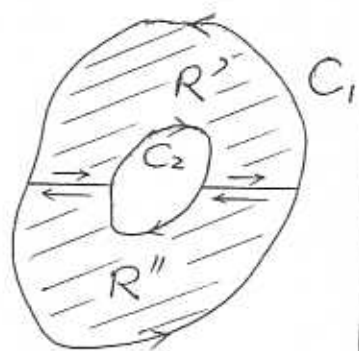
Multiply connected region (hole inside)



16-4-4

Introducing two "cuts", R divided into R' and R'' both simply connected.

- ① outside, curve C_1 , counterclockwise
- ② inside, curve C_2 , clockwise
- ③ R' , R'' boundary curves both counterclockwise.

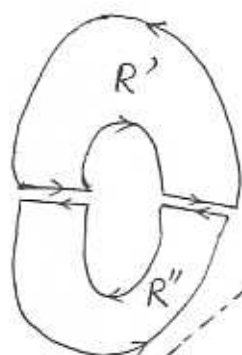


$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_{R'} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

$$+ \iint_{R''} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

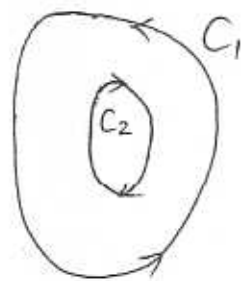
$$\downarrow$$

$$= \oint_{\text{boundary of } R'} f dx + g dy + \oint_{\text{boundary of } R''} f dx + g dy$$



$$= \oint_{C_1} f dx + g dy + \oint_{C_2} f dx + g dy$$

counterclockwise clockwise



Ex. $\oint_C \frac{-y dx + x dy}{x^2 + y^2}$

C is piecewise smooth, counterclockwise closed curve.

a) C does not enclose $(0,0)$

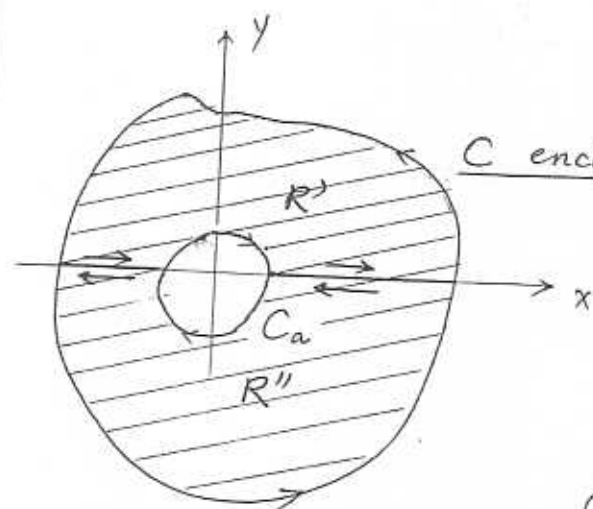
b) C encloses $(0,0)$.

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = \frac{y^2 - x^2}{x^2 + y^2} \quad \text{if } (x,y) \neq (0,0)$$

a) C not enclosing $(0,0)$, R is simply connected
 $\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = 0$, $\oint_C \dots = 0$. G.T. applied.

b) G.T. cannot be directly applied.
 $(0,0)$ is a hole
 R enclosed by C is multiply connected.

16-4-5



C enclosing (0,0)

consider a multiply connected region.

outside boundary C,
counterclockwise
inside boundary C_a ,
clockwise

$$\oint_C + \oint_{C_a} = \oint_{\text{boundary of } R'} + \oint_{\text{boundary of } R''}$$

R' , R'' both simply connected $\rightarrow = 0 + 0 = 0$
 $\iint_{R'} 0 dA = \iint_{R''} 0 dA = 0$ from G.T.

$$\oint_C + \oint_{C_a} = 0 \rightarrow \oint_C = - \oint_{C_a} = \oint_{-C_a}$$

↑
counterclockwise.

Select a special $-C_a$ for easy evaluation
 $-C_a : x^2 + y^2 = a^2$, counterclockwise.

$$\begin{aligned} \oint_{-C_a} \frac{-y dx + x dy}{x^2 + y^2} &= \frac{2}{a^2} \frac{1}{2} \oint_{-C_a} -y dx + x dy \quad \text{area} \\ &= \frac{2}{a^2} \times A = \frac{2}{a^2} \pi a^2 = 2\pi \end{aligned}$$

So, any counterclockwise, closed curve C

$$\rightarrow \oint_C \frac{-y dx + x dy}{x^2 + y^2} = 2\pi$$