Math2111 Introduction to Linear Algebra
Fall 2011
— Midterm Examination (All Sections) —

Name:
Student ID:
Lecture Section:

- There are FOUR questions in this midterm examination.
- Answer all the questions.
- You may write on both sides of the paper if necessary.
- You may use a HKEA approved calculator. Calculators with symbolic calculus functions are not allowed.
- The full mark is 100.

1. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ be the three columns of matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$.

(a) Determine if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. If there is a nontrivial solution, then write down the solution set in parametric vector form. [8]

(b) Is the set of columns of A, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, linearly independent or linearly dependent? [2]

(c) Is the linear system $A\mathbf{x} = \mathbf{b}$ consistent for each vector $\mathbf{b}$ in R^3? If the answer is "yes", explain why. Or if the answer is "no", find out the condition on b_1, b_2, b_3 in $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ for $A\mathbf{x} = \mathbf{b}$ being consistent. [5]

(d) Do the columns of A span R^3? If not, what is the geometric description of $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? [3]

(e) Consider the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. Is it onto R^3? Is it one-to-one? [4]

(f) Is the matrix A invertible? If the answer is "yes", find out A^{-1}. [3]

[25] in total

Solution:

(a) Row operations on the augmented matrix yield the reduced echelon form

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$ from which we have $\begin{cases} x_1 = 2t \\ x_2 = -t \\ x_3 = t \end{cases}$ and

$\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} t$. Therefore, there are nontrivial solutions.

(b) Since $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly dependent.

(c) Row operations on the augmented matrix yield

$$\begin{bmatrix} 1 & 4 & 2 & b_1 \\ 2 & 5 & 1 & b_2 \\ 3 & 6 & 0 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & b_1 \\ 0 & -3 & -3 & b_2 - 2b_1 \\ 0 & -6 & -6 & b_3 - 3b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & b_1 \\ 0 & -3 & -3 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix},$$ from which we see that the system is not consistent for each $\mathbf{b}$. The condition for the system to be consistent is $b_3 - 2b_2 + b_1 = 0$.

(d) No, the columns of A don't span R^3. $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is geometrically a plane according to the consistency condition $b_3 - 2b_2 + b_1 = 0$ found in (c).

(e) It is not onto R^3 as the system $A\mathbf{x} = \mathbf{b}$ is not always consistent. It is not one-to-one either as there are nontrivial solutions for $A\mathbf{x} = \mathbf{0}$.

(f) The matrix A is not invertible.

2. Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5\ \mathbf{a}_6]$ be a 4×6 matrix and the columns of *A* span R^4.

(a) How many pivot positions are there in *A*? [4]

(b) Is the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\}$ linearly independent or linearly dependent? [4]

(c) Does the homogeneous system $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? If the answer is "yes", then find out the number of free variables in the solution set. [5]

(d) For each vector $\mathbf{b}$ in R^4, is the linear system $A\mathbf{x} = \mathbf{b}$ always consistent? [4]

(e) For the matrix transformation *T* defined by $\mathbf{x} \mapsto T(\mathbf{x}) = A\mathbf{x}$, what is the domain and what is the codomain? [4]

(f) Is the matrix transformation *T* one-to-one? Is *T* onto the codomain? [4]

[25] in total

Solution:

(a) Four pivot positions (one in each row).

(b) Linearly dependent because there are more columns than rows ($6 > 4$).

(c) Yes. Two free variables from $6 - 4 = 2$.

(d) Yes because there is a pivot position in each row.

(e) The domain is R^6 while the codomain is R^4.

(f) The transformation is not one-to-one because there are free variables. It is onto the codomain because $A\mathbf{x} = \mathbf{b}$ is always consistent.

3.

(a) Find the inverse (if it exists) of the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. [15]

(b) Consider the linear system $\begin{cases} x_2 - 2x_3 + x_4 = b_1 \\ x_1 - x_2 + 2x_3 - x_4 = b_2 \\ -2x_1 + 2x_2 - 2x_3 + x_4 = b_3 \\ x_1 - x_2 + x_3 = b_4 \end{cases}$.

Solve the system by expressing (x_1, x_2, x_3, x_4) in terms of (b_1, b_2, b_3, b_4). [10]

[25] in total

Solution: (a) Perform row operations to the combined matrix $[\,A \mid I_4\,]$:

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-r_1+r_2} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-r_2+r_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2r_3+r_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 2 & -2 & 1 \end{array}\right] \xrightarrow{-r_4+r_2} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[-r_2+r_1]{r_3 \leftrightarrow r_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -2 & 2 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \end{array}\right].$$

Hence A is invertible, and A^{-1} is given by:

$$A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 1 & -1 & 2 & -1 \\ -2 & 2 & -2 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}.$$

(b) The linear system is actually the same as $A^{-1}\mathbf{x} = \mathbf{b}$. So it has a unique solution $\mathbf{x}$ given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A\mathbf{b} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} b_1 + b_2 \\ b_1 + 2b_2 + b_3 \\ b_2 + b_3 + b_4 \\ b_3 + 2b_4 \end{bmatrix}.$$

4.

(a) Find $\det A$ in terms of k for $A = \begin{bmatrix} 1 & 4 & 1 & 2 \\ -1 & -2 & 1 & 1 \\ -2 & 0 & 3 & k \\ 1 & 2 & k & 5 \end{bmatrix}$. [15]

(b) Let $k = 3$. Find $\det(-2A^{-1})$ if A is invertible. [5]

(c) Let $k = 3$. Find $\det(A^3 A^T)$. [5]

[25] in total

Solution: (a) By suitable row replacement operations, we can transform A to:

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ -1 & -2 & 1 & 1 \\ -2 & 0 & 3 & k \\ 1 & 2 & k & 5 \end{bmatrix} \xrightarrow[\substack{2r_1+r_3 \\ -r_1+r_4}]{r_1+r_2} \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 2 & 2 & 3 \\ 0 & 8 & 5 & k+4 \\ 0 & -2 & k-1 & 3 \end{bmatrix} \xrightarrow[-4r_2+r_3]{r_2+r_4} \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & -3 & k-8 \\ 0 & 0 & k+1 & 6 \end{bmatrix}.$$

Since only row replacements are used, the matrices will have the same determinants. Then by cofactor expansion:

$$\begin{aligned} \det A &= \det \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & -3 & k-8 \\ 0 & 0 & k+1 & 6 \end{bmatrix} = 1 \cdot 2 \cdot \det \begin{bmatrix} -3 & k-8 \\ k+1 & 6 \end{bmatrix} \\ &= 1 \cdot 2 \cdot \{-18 - (k+1)(k-8)\} = -2(k-5)(k-2). \end{aligned}$$

(b) For $k = 3$, $\det A \neq 0$ and A is invertible.

$$\det(-2A^{-1}) = (-2)^4 \det(A^{-1}) = \frac{16}{\det A} = \frac{16}{4} = 4.$$

(c) $\det(A^3 A^T) = \det(A^3)\det(A^T) = (\det A)^4 = 4^4 = 256$.