Chapter 1

Dynamical system: Pendulum

1.1 Notations Used In This Chapter

ω	Angular frequency
ϕ	Phase shift
A	Amplitude
l	Length of string
T	Period of Oscillation

1.2 Derivation of the Governing equation

From the knowledge we learnt in high school, motion of system, in general, are followed Newton's second law,

$$F = ma.$$

In order to derive the governing equation of the motion of pendulum, we should know what are F and a in this equation.

Observe the motion of pendulum, this is no doubt that the locus of pendulum is an arc. Depends on the angular displacement of the pendulum, it is

$$s = l\theta,$$

where θ is the angular displacement of pendulum from the initial position. Therefore the velocity and acceleration of pendulum are

$$v = \frac{ds}{dt} = l\frac{d\theta}{dt}, \qquad a = \frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}. \tag{1.1}$$

The second thing we should know is the expression of F. Observe fig.1.1, by decomposition of the downward force acting on pendulum, the force components can be found. Since the component along the string cancel with the tension force

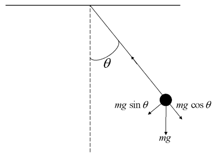


Figure 1.1: Forces acting on the mass.

of the string, the force only affecting the motion is the component perpendicular to the string. Therefore

$$F = -mg\sin\theta. \tag{1.2}$$

The reason of the minus sign of F because θ is decreasing due to acceleration by assuming the center dotted line is 0^o. Combine (1.1) and (1.2), the motion of pendulum can be written as

$$-mg\sin\theta = ml\frac{d^2\theta}{dt^2},$$
$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{1.3}$$

This is the governing equation of undamped, unforced pendulum. Moreover, since the presence of $\sin\theta$, (1.3) is a non-linear equation.

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

However, in case of small-oscillation, $\sin\theta \simeq \theta$, therefore, the governing equation of small oscillation is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0. \tag{1.4}$$

This is also the well known equation of simple harmonic oscillation (SHO), which the pendulum is undergoing simple harmonic motion (SHM).

In fact, we can derive (1.3) from another direction. Considering the polar coordinate system, such that

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

This implies

$$r = \sqrt{x^2 + y^2}\,, \qquad \theta = \tan^{-1}\frac{y}{x}.$$

In fig. 1.2, $\hat{r} = \hat{r}(\theta)$ and $\hat{\theta} = \hat{\theta}(\theta)$. Moreover, they can be expressed in terms of

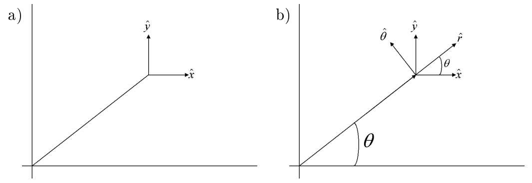


Figure 1.2: a) The setting of Cartesian coordinate system; b) The setting of Polar coordinate system.

x, y and θ, such that

$$\begin{aligned} \hat{a} &= \hat{x}\cos\theta + \hat{y}\sin\theta, \\ \hat{\theta} &= -\hat{x}\sin\theta + \hat{y}\cos\theta, \\ |\hat{r}| &= |\hat{\theta}| = 1. \end{aligned}$$

We write $\vec{F} = m\frac{d^2\vec{X}}{dt^2}$, while the cartesian coordinate components of $\vec{F}$ are

$$F_x = m\frac{d^2x}{dt^2} \qquad F_y = m\frac{d^2y}{dt^2}.$$

In polar coordinates, $\vec{X} = r\hat{r}$. Therefore the corresponding velocity of $\vec{X}$ is

$$\frac{d\vec{X}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}. \tag{1.5}$$

In (1.5), $\frac{d\hat{r}}{d\theta}$ equals to

$$\frac{d\hat{r}}{d\theta} = -\hat{x}\sin\theta + \hat{y}\cos\theta = \dot{\theta},$$

so (1.5) can be further simplified to

$$\frac{d\vec{X}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta},\ \dot{r} = \frac{dr}{dt},\ \dot{\theta} = \frac{d\theta}{dt}. \tag{1.6}$$

Furthermore, the acceleration is

$$\frac{d^2\vec{X}}{dt^2} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{d\theta}\dot{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{d\theta}\dot{\theta},$$

substitute this,

$$\frac{d\hat{\theta}}{d\theta} = -\hat{x}\cos\theta - \hat{y}\sin\theta = -\hat{r},$$

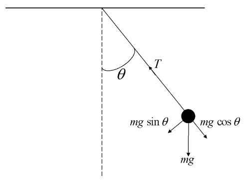


Figure 1.3: Forces acting of the mass with tension force on the string.

so

$$\frac{d^2\vec{X}}{dt^2} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}. \tag{1.7}$$

Here, $\left(\ddot{r} - r\dot{\theta}^2\right)$ and $\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$ are the centripetal acceleration and the coridis acceleration respectively. Thereofer, $\vec{F}$ can be written in polar coordinate with the expression

$$F = F_r\hat{r} + F_\theta\hat{\theta},$$

and the components of F are

$$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right) \qquad F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right).$$

Consider the forces in fig. 1.3, their relationships can be written as

$$mg\cos\theta - T = m\left(\ddot{r} - r\dot{\theta}^2\right) \tag{1.8}$$

$$-mg\sin\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) \tag{1.9}$$

T in (1.8) is tension. For a rob or string without elasticity, $r = l$ and $\dot{r} = \ddot{r} = 0$. Therefore

$$T = mg\sin\theta + ml\dot{\theta}^2, \tag{1.10}$$

$$ml\ddot{\theta} = -mg\sin\theta \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{1.11}$$

This is the same as we obtained by the previous approach.

In case of the string is a spring, it satisfies Hooke's Law, $F = -k(x - l)$, where l is the length of unstretched spring, so

$$T = k(r - l)$$

$$\ddot{r} = r\dot{\theta}^2 + g\cos\theta - \frac{k}{m}(r - l) \tag{1.12}$$

$$\ddot{\theta} = -\frac{g}{r}\sin\theta - 2\frac{\dot{r}\dot{\theta}}{r} \tag{1.13}$$

This are the governing equations for a spring pendulum, however, this equations will not be discussed in this chapter.

1.3 Physical quantities related to pendulum motion

1.3.1 Small amplitude of oscillation

Amplitude and Period

Since (1.4) is a second order ODE, it can be solve by using the auxiliary equation, and its solution is

$$\theta(t) = a\cos\omega t + b\sin\omega t = A\cos(\omega t + \phi), \tag{1.14}$$

where $\omega = \sqrt{\frac{g}{l}}$. The period of such oscillation can be obtained by considering $\theta(t+T) = \theta(t)$. In this expression, t is time and T is period (this is different from the previous T). Since $\theta(t)$ is written as (1.14), T can be calculated and it should be $\frac{2\pi}{\omega}$.

$$T = \frac{2\pi}{\omega}. \tag{1.15}$$

Now, let us consider the initial value problem of pendulum. The initial conditions are

$$\theta(0) = \theta_o, \qquad \dot\theta(0) = U_o.$$

Follow previous works, θ_o and $\dot\theta_o$ can be expressed as

$$\theta_o = A\cos\phi \qquad U_o = -\omega A\sin\phi.$$

Rewrite these equations, so

$$\begin{aligned}
\theta_o^2 + \left(\frac{U_o}{\omega}\right) &= A^2\cos^2\phi + A^2\sin^2\phi \\
&= A^2 \\
A &= \sqrt{\theta_o^2 + \frac{U_o}{\omega}}.
\end{aligned} \tag{1.16}$$

$$\begin{aligned}
\frac{-\omega A\sin\phi}{A\cos\phi} &= \frac{U_o}{\theta_o} \\
\tan\phi &= -\frac{U_o}{\omega\theta_o} \\
\phi &= \tan^{-1}\frac{-U_o}{\omega\theta_o}.
\end{aligned} \tag{1.17}$$

Example

1.

$$\begin{aligned}
\theta(0) &= \theta_o,\ \dot\theta(0) = 0 \ \text{ (with angle } \theta_o \text{ and without push)} \\
\theta(t) &= \theta_o\cos(\omega t)
\end{aligned}$$

2.

$$\begin{aligned}
\theta(0) &= 0,\ \dot\theta(0) = U_o \ \text{ (at bottom initially and give a push)} \\
\theta(t) &= \frac{U_o}{\omega}\sin(\omega t) \qquad \{\phi = \frac{\pi}{2}\}
\end{aligned}$$

Conservation of Energy

In order to study the conservation of energy, we should go back to (1.14). The first derivative of this is

$$\dot\theta(t) = -\omega A\sin(\omega t + \phi). \tag{1.18}$$

Square (1.14) and (1.18) and add them together,

$$\omega^2\theta^2 + \dot\theta^2 = \omega^2 A^2. \tag{1.19}$$

This is noticed that (1.19) is independent of time t, the first term $\omega^2\theta^2$ is proportion to potential energy and the second term $\dot\theta^2$ is proportion to kinetic energy.

Consider (1.14) again, by writing $\dot\theta = u$, this implies $\dot u = -\omega^2\theta$.

$$\begin{aligned}
\omega^2\theta\dot\theta + u\dot u &= \omega^2\theta u - \omega^2\theta u \\
&= 0 \\
\omega^2\frac{1}{2}\frac{d}{dt}\theta^2 + \frac{1}{2}\frac{d}{dt}u^2 &= 0 \\
\frac{d}{dt}\left(\omega^2\theta^2 + u^2\right) &= 0.
\end{aligned}$$

This is the change of total energy. With it equals zero, this means that energy is conserved and can be expressed as

$$\omega^2\theta^2 + u^2 = \omega^2 A = \text{ constant}. \tag{1.20}$$

1.3.2 Large amplitude of oscillation

Conservation of Energy

What will happen if the restriction of **small** amplitude is removed? If this restriction is removed, (1.4) is no longer be valid to this problem, and (1.3) should be consider instead. Similar as previous sections, we write the velocity as following

$$\dot\theta = u \qquad \dot u = -\omega^2\sin\theta.$$

Substitute these into (1.3), then

$$\begin{aligned} \omega^2 \sin\theta\dot{\theta} + u\dot{u} &= 0 \\ -\omega^2 \frac{d}{dt}\cos\theta + \frac{1}{2}\frac{d}{dt}u^2 &= 0 \\ \frac{d}{dt}\left(-\omega^2\cos\theta + \frac{1}{2}u^2\right) &= 0 \qquad (1.21) \end{aligned}$$

$$\frac{1}{2}u^2 - \omega^2\cos\theta = -\omega^2\cos\theta_m = E, \qquad (1.22)$$

(1.21) is called the integral of motion. In (1.22), θ_m is defined to be the maximum value of $\theta(t)$ and E is constant. The same situation as small amplitude oscillation, the first derivative of this equals to zero, that means for large amplitude oscillation, energy is still conserved. In fact, this is a must for any physical system, if we assume energy is represented by kinetic energy and potential energy. Therefore, if one is decreasing, another one should be increasing while their sum should be a constant.

In general, conservation of energy is a good tool for checking the correctness of the governing equation, or later, the numerical methods and programs.

Period

What is the difference of the period between small and large oscillation? This question can be answered by following (1.22).

$$\begin{aligned} u^2 &= 2\omega^2(\cos\theta - \cos\theta_m) \\ \frac{d\theta}{dt} &= \sqrt{2}\,(\cos\theta - \cos\theta_m)^{\frac{1}{2}} \\ dt &= \frac{1}{\sqrt{2}\omega\,(\cos\theta - \cos\theta_m)^{\frac{1}{2}}}\,d\theta \\ \int_0^t dt &= \frac{1}{\sqrt{2}\omega}\int_\theta^{\theta_o}\frac{d\theta}{(\cos\theta - \cos\theta_m)^{\frac{1}{2}}} \\ t &= \frac{1}{\sqrt{2}\omega}\int_\theta^{\theta_m}\frac{d\theta}{(\cos\theta - \cos\theta_m)^{\frac{1}{2}}}. \qquad (1.23) \end{aligned}$$

This is the time for pendulum move from θ_0 to θ and (1.23) is in terms of Jocabi elliptic function. The value of θ_m will affect the value of t.

As stated before, for small amplitude oscillation, ($\theta_m \ll 1$), period $T = \frac{2\pi}{\omega}$. However, what we happen if θ_m increases near to 1?

For simple pendulum, period of its motion is T, therefore, the time of going from bottom to θ_m should be $\frac{T}{4}$ (fig. 1.4). Therefore

$$T = \frac{4}{\sqrt{2}\omega}\int_0^{\theta_m}\frac{1}{(\cos\theta - \cos\theta_m)^{\frac{1}{2}}}\,d\theta \qquad (1.24)$$

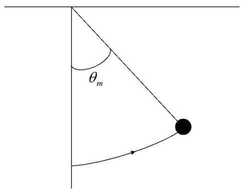


Figure 1.4: Motion of pendulum with θ_m displacement.

By using these expressions

$$\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta), \qquad \cos\theta = 1 - 2\sin^2\frac{\theta}{2},$$

$\cos\theta - \cos\theta_m$ can be re-written as this

$$\cos\theta - \cos\theta_m = 2\left(\sin^2\frac{\theta_m}{2} - \sin^2\frac{\theta}{2}\right). \qquad (1.25)$$

Substitute (1.25) into (1.24),

$$T = \frac{2}{\omega}\int_0^{\theta_m}\frac{1}{\left(\sin^2\frac{\theta_m}{2} - \sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}}\,d\theta. \qquad (1.26)$$

Define $a = \sin\frac{\theta_m}{2}$, such that a is a constant. Therefore (1.26) can be further simplified to

$$T = \frac{2}{\omega}\int_0^{\theta_m}\frac{1}{a\left(1 - \frac{1}{a^2}\sin^2\theta 2\right)^{\frac{1}{2}}}\,d\theta. \qquad (1.27)$$

We let $\sin\phi = \frac{1}{a}\sin\frac{\theta}{2}$. If $0 \le \theta \le \theta_m$, so $0 \ge \sin\phi \le 1$ and this implies $0 \le \phi \le \frac{\pi}{2}$. Moreover,

$$\begin{aligned} \cos\phi\,d\phi &= \frac{1}{2a}\cos\frac{\theta}{2}\,d\theta \\ d\theta &= \frac{2a\cos\phi}{\cos\frac{\theta}{2}}\,d\phi \\ &= \frac{2a\cos\phi}{\left(1 - a^2\sin^2\phi\right)^{\frac{1}{2}}}\,d\phi \qquad (1.28) \end{aligned}$$

Now, we have the range of ϕ and rewrite $d\theta$ in terms of ϕ, therefore, we can take more steps for simplified (1.25).

$$T = \frac{2}{\omega}\int_0^{\frac{\pi}{2}}\frac{2a\cos\phi}{\left(1 - a^2\sin^2\phi\right)^{\frac{1}{2}}}\frac{1}{a\cos\phi}\,d\phi$$

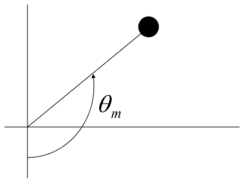


Figure 1.5: Another figure of motion of pendulum with θ_m displacement.

$$
\begin{aligned}
&= \frac{4}{\omega}\int_0^{\frac{\pi}{2}} \frac{1}{\left(1-a^2\sin^2\phi\right)^{\frac{1}{2}}}d\phi \\
k(a) &= \int_0^{\frac{\pi}{2}} \frac{1}{\left(1-a^2\sin^2\phi\right)^{\frac{1}{2}}}d\theta \qquad (1.29)\\
T &= \frac{4}{\omega}k(a). \qquad (1.30)
\end{aligned}
$$

(1.29) is call the first kind complete elliptic integral. One more reminding is the meaning of notation, θ_m is the maximum value of θ, that means the amplitude and $a = sin\frac{\theta_m}{2}$.

As a tends to 0, T will tend to $\frac{2\pi}{\omega}$, once again, it agree with the period of small amplitude oscillation before hand. Let's turn back to the question, what would happen if θ_m is getting larger? In case if $\theta_m = \pi$, so $a = 1$, T will tend to infinity.

How does T diverges at $\theta \simeq \pi$? Consider θ_m close to π (fig. 1.5), since, θ_m is large, the relationship $\sin\theta \sim \theta$ does not hold in this case, therefore, (1.3) should be considered. Let $\theta = \pi - \phi$, for ϕ is also a function of t, so

$$
\begin{aligned}
\sin\theta &= \sin(\pi-\phi)\\
&= \sin\pi\cos\phi + \cos\pi\sin(-\phi)\\
&= \sin\phi,\\
\frac{d^2\theta}{dt^2} &= -\frac{d^2\phi}{dt^2}.
\end{aligned}
$$

Due to these changes, the following can be found

$$
\begin{aligned}
&-\ddot{\phi} + \omega^2\sin\phi = 0\\
&\Rightarrow \ddot{\phi} - \omega^2\sin\phi = 0. \qquad (1.31)
\end{aligned}
$$

Since ϕ is very small, therefore $\sin\phi \simeq \phi$, and (1.31) becomes

$$\ddot{\phi} - \omega^2\phi = 0, \qquad (1.32)$$

and its solution is such that C and D are constants,

$$\phi(t) = Ce^{\omega t} + De^{-\omega t}. \qquad (1.33)$$

Suppose the initial conditions are:

$$
\begin{aligned}
\theta(0) &= \theta_m = \pi - \phi(0) = \pi - \phi_o\\
\dot{\theta}(0) &= 0 = -\dot{\phi}(0) = -\dot{\phi}_o.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\phi_o &= C + D\\
\dot{\phi}_o &= 0 = \omega C - \omega D
\end{aligned}
$$

$$C = D = \frac{1}{2}\phi_o$$

$$\phi(t) = \frac{1}{2}\phi_o\left(e^{\omega t} + e^{-\omega t}\right). \qquad (1.34)$$

Since $\theta = \pi - \phi$,

$$\theta(t) = \pi - \frac{1}{2}(\pi - \theta_m)\left(e^{\omega t} + e^{-\omega t}\right). \qquad (1.35)$$

We can see from (1.35), when $\theta_m = \pi$,

$$\theta(t) = \pi, \qquad (1.36)$$

which is independent of t, this means that if the initial position of pendulum is at the top, the weight will not move. You can either say it period is zero or is infinity. However, this still does not answer how does T diverges. By using (1.35) and the setting as fig. 1.6, this question can be answered. To answer this question, we first calculate the time S for pendulum moving from θ_m to θ'_m. Using (1.35),

$$\theta(S) = \theta'_m = \pi - \frac{1}{2}(\pi - \theta_m)\left(e^{\omega S} + e^{-\omega S}\right). \qquad (1.37)$$

Assume, $\omega S \gg 1$, so

$$\theta'_m \simeq \pi - \frac{1}{2}(\pi - \theta_m)e^{\omega S}. \qquad (1.38)$$

Continue the algebraic works,

$$
\begin{aligned}
e^{\omega S} &= 2\frac{\pi - \theta'_m}{\pi - \theta_m}, \qquad (1.39)\\
S &= \frac{1}{\omega}\ln\frac{2(\pi - \theta'_m)}{\pi - \theta_m}. \qquad (1.40)
\end{aligned}
$$

If the period of the motion for $\theta_o = \theta'_m$ is denoted by T' and the period for the case of $\theta_o = \theta_m$ by T, then

$$T' = 4S + T. \qquad (1.41)$$

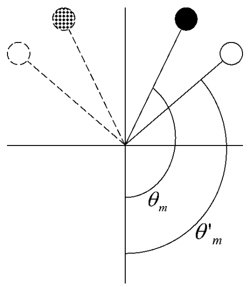


Figure 1.6: Motion of pendulum from the position θ_m to position θ'_m in the time S.

Since θ'_m is fixed and not equals to π, therefore T' is a finite number. Combine (1.41) and (1.40),

$$\begin{aligned} T &= \frac{4}{\omega}\left[\ln 2 + \ln(\pi - \theta'_m) - \ln(\pi - \theta_m)\right] + T' \\ &= -\frac{4}{\omega}\ln(\pi - \theta_m) + T' \\ &= -\frac{4}{\omega}\ln\left(1 - \frac{\theta_m}{\pi}\right) + T'. \end{aligned} \tag{1.42}$$

If θ'_m is decreased to 0, so

$$T = -\frac{4}{\omega}\ln\left(1 - \frac{\theta_m}{\pi}\right) \tag{1.43}$$

We can see clearly that if $\theta_m = \pi$, T will tend to infinity as the presence of $\ln 0$. Up to now, we have two methods to express the period of large oscillation, however, both of them consist of two variables, ω and θ_m. Moreover, the dimensions in these equations are meaningless. Such of these things can be simplified by using a technique call non-dimensionalization.

Non-dimensionalization

In short, non-dimensionalization is a process taking away dimension from equation. For further study, there is a topic call dimension analysis, however, this will not be too deep here.

Since ω is frequency, therefore its unit is

$$[\omega] = t^{-1}.$$

In order to make time becomes dimensionless, we choose $\hat{t}$ be the non-dimensional time and denoted it as

$$\hat{t} = t\frac{\omega}{2\pi}. \tag{1.44}$$

For small oscillation, T equals to $\frac{\omega}{2\pi}$. therefore, $\hat{T} = 1$. If t is transformed, the equation should be updated,

$$\frac{d\theta}{dt} = \frac{d\hat{t}}{dt}\frac{d\theta}{d\hat{t}} = \frac{\omega}{2\pi}\frac{d\theta}{d\hat{t}}, \tag{1.45}$$

$$\frac{d^2\theta}{dt^2} = \frac{d\hat{t}}{dt}\frac{d}{d\hat{t}}\left(\frac{d\theta}{d\hat{t}}\right) = \left(\frac{\omega}{2\pi}\right)^2\frac{d^2\theta}{d\hat{t}^2}, \tag{1.46}$$

$$\begin{aligned} \left(\frac{\omega}{2\pi}\right)^2\frac{d^2\theta}{d\hat{t}^2} + \omega^2\sin\theta &= 0 \\ \frac{d^2\theta}{d\hat{t}^2} + (2\pi)^2\sin\theta &= 0. \end{aligned} \tag{1.47}$$

After the non-dimensionalization, (1.47) becomes the non-dimensional equation for pendulum motion.

Apply the same technique to (1.30) and (1.43), they become

$$(1.30) \Rightarrow \hat{T} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - a^2\sin^2\phi}}d\phi, \tag{1.48}$$

$$(1.42) \Rightarrow \hat{T} = -\frac{2}{\pi}\ln\left(1 - \frac{\theta_m}{\pi}\right). \tag{1.49}$$

We call (1.49) the true solution while (1.48) the approximate solution. There are various methods finding the approximate solution, the first method is by Taylor series expansion.

$$\frac{1}{\sqrt{1 - a^2\sin^2\phi}} \simeq 1 + \frac{1}{2}a^2\sin^2\phi + O(a^4)$$

$$\begin{aligned} T &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left[1 + \frac{1}{2}a^2\sin^2\phi + O(a^4)\right]d\phi \\ &= \frac{2}{\pi}\left[\frac{\pi}{2} + \frac{1}{2}a^2\frac{\pi}{4} + O(a^4)\right] \qquad \{a^2 = \sin^2\frac{\theta_m}{2} \sim \frac{\theta_m^2}{4} + O(\theta_m^4)\} \\ &= 1 + \frac{\theta_m^2}{16} + O(\theta_m^4). \end{aligned} \tag{1.50}$$

The second method of solving (1.48) is integrating this directly. This can be done by using Rouberg integration. The third method of finding period of pendulum motion is solving the ODE directly. Recall (1.47),

$$\ddot{\theta} + (2\pi)^2\sin\theta = 0.$$

Decompose it by these

$$\dot{\theta} = u, \qquad \dot{u} = -(2\pi)^2 \sin\theta.$$

The initial conditions are

$$\theta(0) = -|\theta_m|, \qquad \dot{\theta}(0) = 0.$$

The goal is finding time t_*, such that $\theta(t_*) = 0$. Therefore period $T = 4t_*$. Remind once more, T, t and t_* are all dimensionless for the following unless further notification.

The idea of this method is finding time t_f such that θ just becomes positive. However, before finding this, δt should be obtained, such that $t_* = t_f - \delta t$. We consider the situation that pendulum is first resting at the bottom, and then move to one of one direction. At time δt, the position is θ_f, and its velocity is $\dot{\theta}(\delta t) = \dot{\theta}_f$. Therefore, we can treat the motion be a small oscillation, and the following equation will be solved instead of (1.47).

$$\ddot{\theta} + (2\pi)^2 \theta = 0 \tag{1.51}$$

After solving (1.51),

$$\theta(t) = A \sin(2\pi t). \tag{1.52}$$

Then a system of equations is obtained,

$$\begin{aligned} A \sin(2\pi\delta t) &= \theta_f \\ 2\pi A \cos(2\pi\delta t) &= \dot{\theta}_f. \end{aligned}$$

Solve this again,

$$\begin{aligned} \frac{1}{2\pi}\tan(2\pi\delta t) &= \frac{\theta_f}{\dot{\theta}_f} \\ \delta t &= \frac{1}{2\pi}\tan^{-1}\left(\frac{2\pi\theta_f}{\dot{\theta}_f}\right). \end{aligned} \tag{1.53}$$

Hence, t_* can be found.

1.4 Pendulum with Friction

Up to now, we discussed those about frictionless pendulum motion. In the following chapter, pendulum with friction will be discussed. We define the model of friction be

$$F_f = F_f(V), \tag{1.54}$$

such that $F_f(V) \propto -V$. This means that friction force is opposited to the motion and assume it is linear in velocity. Then, we write it in this form

$$F_f = a + bV + cV^2 + \cdots. \tag{1.55}$$

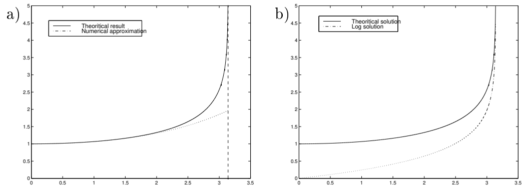


Figure 1.7: (a) For small θ_m, the period T close to 1. Increasing the amplitude, increases the period of the oscillation. If we increase θ_m further close to π, the period tends to infinity, representing a stationary state of the pendulum. (b) For θ_m tends to π, the solution has asymptote with (1.49).

If the object does not has motion, $V = 0$, therefore,

$$F_f(V = 0) = 0 \quad \Rightarrow \quad a = 0.$$

Moreover, we assume the leading term b is dominate, so

$$F_f(V) = bV. \tag{1.56}$$

Since $V = \dot{\theta}$, therefore, the equation of pendulum motion with friction is

$$\ddot{\theta} + \lambda\dot{\theta} + \omega^2 \sin\theta = 0. \tag{1.57}$$

In (1.57), $\lambda\dot{\theta}$ is the friction term, that means it represents friction force.

Now, as study the motion of frictionless pendulum, we assume small oscillation of pendulum first, such as $\sin\theta \simeq \theta$. Therefore, the equation will become

$$\ddot{\theta} + \lambda\dot{\theta} + \omega^2\theta = 0. \tag{1.58}$$

The auxiliary equation of (1.58) is

$$\alpha^2 + \lambda\alpha + \omega^2 = 0. \tag{1.59}$$

It is just a simple quadratic equation. The solution of α are

$$\alpha_\pm = -\frac{1}{2}\lambda \pm \frac{1}{2}\sqrt{\lambda^2 - 4\omega^2}. \tag{1.60}$$

For convenience, we define $\gamma = \frac{\lambda}{2}$, therefore (1.60) becomes

$$\alpha_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega^2}. \tag{1.61}$$

As we learnt in high school, the value of the determinant of (1.60) will affect the nature of root. There are totally three cases for the determinant

1. $\omega^2 > \gamma^2$,
2. $\omega^2 < \gamma^2$,
3. $\omega^2 = \gamma^2$.

For different cases, the solution of (1.58) will be different, this implies that the motion of pendulum will be different also.

Underdamped pendulum, $\omega^2 > \gamma^2$

The first case we study is underdamped pendulum. This will be the case if $\omega^2 > \gamma^2$. We define

$$\omega_1^2 = \omega^2 - \gamma^2, \tag{1.62}$$

therefore ω_1^2 should be negative and its square root is a complex number. Rewrite (1.61) by substituting (1.62) into it, then

$$\alpha_\pm = -\gamma \pm i\omega_1, \quad i^2 = -1, \ \omega_1 \text{ is real.} \tag{1.63}$$

After these, the general solution of θ is known and equals to

$$\theta(t) = e^{-\gamma t}\left[C_1 e^{i\omega_1 t} + C_2 e^{-i\omega^1 t}\right]. \tag{1.64}$$

θ is real, and equals to its complex conjugate. If we define the complex conjugate of θ be θ^*, so

$$\theta = \theta^*. \tag{1.65}$$

Therefore

$$\begin{aligned} C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t} &= C_1^* e^{-i\omega_1 t} + C_2^8 e^{-\omega_1 t} \\ (C_1 - C_2^*)\left(1 - e^{2\omega_1 t}\right) &= 0 \\ C_1 &= C_2^* \quad \forall t. \end{aligned}$$

Base on this, θ can be rewritten as

$$\theta(t) = e^{-\gamma t} 2\Re\left[C_1 e^{i\omega_1 t}\right]. \tag{1.66}$$

Takes $C_1 = \frac{1}{2} A e^{i\phi}$, with A and ϕ are real, so

$$\theta(t) = A e^{-\gamma t}\cos(\omega_1 t + \phi). \tag{1.67}$$

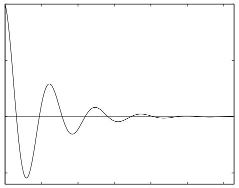


Figure 1.8: The time series of underdamped pendulum.

Overdamped pendulum

The second damping pendulum we discuss is overdamped pendulum. The reason of this is $\omega^2 < \gamma^2$. Therefore, α should be double root and they are

$$\alpha_- = -\gamma_1 = -\gamma - \sqrt{\gamma^2 - \omega^2} < 0, \tag{1.68}$$
$$\alpha_+ = -\gamma_2 = -\gamma + \sqrt{\gamma^2 - \omega^2} < 0. \tag{1.69}$$

The general solution for overdamped pendulum is

$$\theta(t) = C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t}. \tag{1.70}$$

Although $-\gamma_1$ and $-\gamma_2$ are both negative, their magnitude are difference. According to (1.68) and (1.69), γ_1 is greater than γ_2. Therefore, $e^{-\gamma_1 t}$ can be treated as a fastest decaying solution, which affect the motion most just after the beginning. However, for long time decay, it is controlled by the second term, $e^{-\gamma_2 t}$, which is the slowest decaying solution.

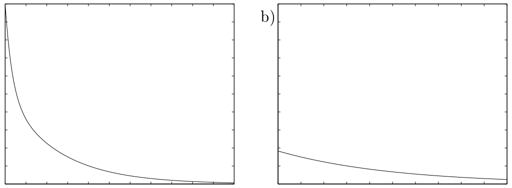
a)


Figure 1.9: a) The global view of the time series of overdamped pendulum; b) the zoom view of the overdamped pendulum for longer time.

As we can see in fig.1.9, it takes long time for pendulum to be rest. This is the same as the explanation as before.

Critically damped pendulum

The last one we study is critically damping. This is due to the situation $\omega^2 = \gamma^2$. Therefore the only solution of α is

$$\alpha_{\pm} = -\gamma. \tag{1.71}$$

From this, we know

$$\theta_1(t) = C_1 e^{-\gamma t}. \tag{1.72}$$

However, (1.57) is a second order ODE, it should has two solution, so we need to find another solution such that it satisfy the initial conditions. The method we use is reduction of order. We assume the second solution is in this form

$$\theta_2(t) = U(t)\,\theta_1(t)\,. \tag{1.73}$$

Now, we rewrite (1.57) by replacing λ and ω by γ, so

$$\ddot{\theta} + 2\gamma\dot{\theta} + \gamma^2\theta = 0. \tag{1.74}$$

We differentiate (1.73) for its first and second derivative. They are

$$\dot{\theta}_2 = \dot{U}\theta_1 + U\dot{\theta}_1 \tag{1.75}$$

$$\ddot{\theta}_2 = \ddot{U}\theta_1 + 2\dot{U}\dot{\theta}_1 + U\ddot{\theta}_1 \tag{1.76}$$

Substitute (1.75) and (1.76) into (1.74)

$$\begin{aligned}
\ddot{U}\theta_1 + 2\dot{U}\dot{\theta}_1 + U\ddot{\theta}_1 + 2\gamma\left(\dot{U}\theta_1 + U\dot{\theta}_1\right) + \gamma^2 U\theta_1 &= 0 \\
\ddot{U}\theta_1 + \dot{U}\left(2\dot{\theta}_1 + 2\gamma\theta_1\right) + U\left(\ddot{\theta}_1 + 2\gamma\dot{\theta}_1 + \gamma^2\theta_1\right) &= 0.
\end{aligned} \tag{1.77}$$

Since $\ddot{\theta}_1 + 2\gamma\dot{\theta}_1 + \gamma^2\theta_1 = 0$ in (1.77) is just the governing equation with θ replaced by θ_1, and θ_1 is one of the solutions, therefore, this expression equals to zero. Moreover

$$2\dot{\theta}_1 = -2\gamma e^{-\gamma t} = -2\gamma\theta_1, \tag{1.78}$$

so $2\dot{\theta}_1 + 2\gamma\theta_1$ in (1.77) also equals to zero. As a result, (1.77) can be simplified and become

$$\ddot{U}\theta_1 = 0. \tag{1.79}$$

Since $\theta_1 \neq 0$, so

$$\ddot{U} = 0 \tag{1.80}$$

$$U = A + Bt \tag{1.81}$$

Recall the definition of θ_2, so

$$\theta_2 = A\theta_1 + Bt\theta_1, \tag{1.82}$$

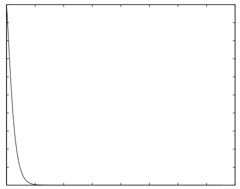


Figure 1.10: The time series of critically damped pendulum.

where $A\theta_1$ is the old solution and the new solution is $Bt\theta_1$. Use the same notation as the previous cases, the general solution for this case is

$$\theta(t) = (C_1 + C_2 t)\, e^{-\gamma t}. \tag{1.83}$$

Fig.1.10 shows the situation for critically damping. The appearance of this looks similar to the case of overdamped. However, it tends to zero much faster than overdamped pendulum do. Compare these three damping pendulum, for fixed ω, critically damping pendulum has $\theta(r)$ approaches to zero the fastest.

1.5 Damped-Driven pendulum

In pervious sections, we have discussed the situation of the motion without any external forces. From this section going on, the forced-driven pendulum will be our main focus, however, we still concentrate on the small oscillation, $\sin\theta \simeq \theta$.

Solution of Damped-Driven pendulum

The governing equation for damped-driven pendulum motion is

$$\ddot{\theta} + \lambda\dot{\theta} + \omega^2\theta = \epsilon F(t)\,, \tag{1.84}$$

$\epsilon F(t)$ is the external force. This is a second order nonhomogeneous equation. The solution for nonhomogeneous equations are the combination of their homogeneous solution plus the particular solution. The homogeneous equation of (1.84) is

$$\ddot{\theta}\lambda\dot{\theta} + \omega^2\theta = 0, \tag{1.85}$$

all of its solutions have been discussed in the previous sections. Our first task in this situation is finding the particular solution of (1.84).

We consider

$$F(t) = \cos(\Omega t), \tag{1.86}$$

which is the periodic driving force. Write (1.84) as

$$\ddot{\theta} + \lambda\dot{\theta} + \omega^2\theta = \epsilon e^{i\Omega t}, \tag{1.87}$$

its solution is in the form

$$\theta(t) = \theta^R(t) + i\theta^I(t). \tag{1.88}$$

Substitute (1.88) into (1.87),

$$\left(\ddot{\theta}^R + \lambda\dot{\theta}^R + \omega\theta^R\right) + i\left(\ddot{\theta}^I + \lambda\dot{\theta}^I + \omega^2\theta^I\right) = \epsilon\left(\cos(\Omega t) + i\sin(\Omega t)\right). \tag{1.89}$$

Equate the real and imaginary parts,

$$\ddot{\theta}^R + \lambda\dot{\theta}^R + \omega^2\theta^R = \epsilon\cos(\Omega t), \tag{1.90}$$
$$\ddot{\theta}^I + \lambda\dot{\theta}^I + \omega^2\theta^I = \epsilon\sin(\Omega t). \tag{1.91}$$

Our desired solution is $\theta^R = \Re(\theta)$.

We let $\theta(t) = Ae^{i\Omega t}$, and substitute it into (1.87),

$$\begin{aligned}-A\Omega^2 + i\lambda A\Omega + \omega^2 A &= \epsilon\\ A\left(\omega^2 - \Omega^2 + i\lambda\Omega\right) &= \epsilon\\ A &= \frac{\epsilon}{\omega^2 - \Omega^2 + i\lambda\Omega}.\end{aligned} \tag{1.92}$$

We write $A = |A|e^{i\beta}$,

$$\begin{aligned}|A| &= \sqrt{AA*}\\ &= \left(\frac{\epsilon^2}{\left(\omega^2 - \Omega^2\right)^2 + \lambda^2\Omega^2}\right)^{\frac{1}{2}}.\end{aligned} \tag{1.93}$$

It is easily to see that

$$\tan\beta = \frac{\Im|A|}{\Re|A|}. \tag{1.94}$$

Therefore,

$$\theta(t) = |A|e^{i(\Omega t + \beta)}, \tag{1.95}$$

so,

$$\theta^R(t) = \frac{\epsilon}{\sqrt{\left(\omega^2 - \Omega^2\right)^2 + \lambda^2\Omega^2}}\cos(\Omega t + \beta). \tag{1.96}$$

As a result, the general solution of (1.87) is

$$\theta(t) = \text{homogeneous solution} + \theta^R(t). \tag{1.97}$$

From previous section, we know that as time going on, the homogeneous solution will tends to zero, therefore the asymptotic solution of θ is governed by its particular solution.

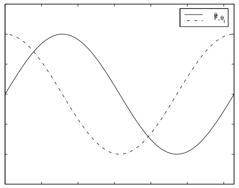


Figure 1.11: This figure shows the phase difference between θ and the exteranl force, F and the velocity θ_t.

Resonance

The basic definition of resonance for pendulum motion is the frequency of external force equals to the natural frequency of pendulum, such as $\omega = \Omega$. If this is so, (1.96) becomes

$$\theta(t) = \frac{\epsilon}{\lambda\Omega}\cos(\Omega t + \beta). \tag{1.98}$$

However, before we going on, we must know what β is.

Recall (1.92), for the case of resonance,

$$\begin{aligned}A &= \frac{\epsilon}{i\lambda\Omega}\\ &= -i\frac{\epsilon}{\lambda\Omega}\\ &= \frac{\epsilon}{\lambda\Omega}e^{\frac{i\pi}{2}}\\ &= |A|e^{i\beta}.\end{aligned}$$

Therefore,

$$\beta = -\frac{\pi}{2}. \tag{1.99}$$

After the knowing of β, (1.98) becomes

$$\begin{aligned}\theta(t) &= \frac{\epsilon}{\lambda\Omega}\cos\left(\Omega t - \frac{\pi}{2}\right)\\ &= \frac{\epsilon}{\lambda\Omega}\sin(\Omega t).\end{aligned} \tag{1.100}$$

The second thing w have to know is when the amplitude is maximum. This can be known by finding the value of Ω such that $|A|$ is maximum with fixed ϵ, λ, ω.

In order to find the maximum amplitude, we have to consider its first derivative,

$$\frac{d}{d\Omega}\left[\frac{\epsilon}{\sqrt{\left(\omega^2-\Omega^2\right)^2+\lambda^2\Omega^2}}\right]=0. \tag{1.101}$$

However, we recommand another format for consideration,

$$\frac{d}{d\Omega}\left[\left(\omega^2-\Omega^2\right)^2+\lambda^2\Omega^2\right]=0. \tag{1.102}$$

This will become

$$\begin{aligned} 2\left(\omega^2-\Omega^2\right)(-2\Omega)+2\lambda^2\Omega &= 0\\ \Omega\left[\lambda^2-2\left(\omega^2-\Omega^2\right)\right] &= 0. \end{aligned}$$

Therefore, solutions are $\Omega=0$ or

$$\begin{aligned} 2\left(\omega^2-\Omega^2\right) &= \lambda^2\\ \Omega^2 &= \omega^2-\frac{\lambda^2}{2}\\ \Omega &= \pm\left(\omega^2-\frac{\lambda^2}{2}\right)^{\frac{1}{2}}. \end{aligned} \tag{1.103}$$

The second derivative of A is

$$\begin{aligned} A''(\Omega) &= 2\left(2\Omega^2-2\omega^2+\lambda^2\right)+2\Omega(4\Omega)\\ &= 12\Omega^2-4\omega^2+2\lambda^2. \end{aligned} \tag{1.104}$$

Now, we substitute the two solutions into A'':

$$A''(0)=-4\omega^2+2\lambda^2\begin{cases} <0 & \text{if } \lambda^2<2\omega^2\\ >0 & \text{if } \lambda^2>2\omega^2 \end{cases}. \tag{1.105}$$

$$\begin{aligned} A''\left(\Omega^2=\omega^2-\frac{1}{2}\lambda^2\right) &= 12\left(\omega^2=\frac{1}{2}\lambda^2\right)-4\omega^2+2\lambda^2\\ &= 8\omega^2-4\lambda^2\\ &= \begin{cases} <0 & \text{if } \lambda^2>2\omega^2\\ >0 & \text{if } \lambda^2<2\omega^2 \end{cases}. \end{aligned} \tag{1.106}$$

In conclusion, if $\lambda^2<2\omega^2$, then $\Omega^2=\omega^2-\frac{1}{2}\lambda^2$ gives maximum amplitude, on the other hand, if $\lambda^2>2\omega^2$, then $\Omega^2=0$ will be the value for maximum amplitude.

We now look at the undamped, forced pendulum at resonance ($\lambda=0$, $\Omega=\omega$). Consider (1.87) again, however, with the change of parameters, it becomes

$$\ddot{\theta}+\omega^2\theta=\epsilon e^{i\omega t}. \tag{1.107}$$

The solution for its homogeneous equation is

$$\theta(t)=e^{\pm i\omega t}. \tag{1.108}$$

We need to find out its particular solution. We let $\theta=U(t)e^{i\omega t}$, then

$$\dot{\theta} = \left(\dot{U}+i\omega U\right)e^{i\omega t}, \tag{1.109}$$

$$\ddot{\theta} = \left[\left(\ddot{U}+i\omega\dot{U}\right)+i\omega\left(\dot{U}+i\omega U\right)\right]e^{i\omega t}. \tag{1.110}$$

After the substitution and cancellation, the following equation can be written

$$\ddot{U}+2i\omega\dot{U}=\epsilon. \tag{1.111}$$

In fact, there is no direct method for finding the solution U for (1.111). However, we can guess the solution. Since we need to find any U that satisfy this equation, except U is a constant, therefore, we take

$$U=\frac{t\epsilon}{2i\omega}.$$

So, the particular solution is

$$\begin{aligned} \theta_p(t) &= \Re\left\{\frac{t\epsilon}{2i\omega}e^{i\omega t}\right\}\\ &= \frac{t\epsilon}{2\omega}\sin(\omega t). \end{aligned} \tag{1.112}$$

Combine the homogeneous solution and the particular solution, the general solution of θ is

$$\theta(t)=A\cos(\omega t)+B\sin(\omega t)+\frac{\epsilon}{2\omega}t\sin(\omega t). \tag{1.113}$$

Observe this general solution, we find that the particular solution is an unbounded solution. Although $\sin(\omega t)$ is a bounded function, the multipiler t will increase as time goes by. Therefore, this particular solution is an unbounded solution. Moreover, as our first assumption, this solution only valid for small oscillations ($\sin\theta\sim\theta$). For large oscillation, we have to consider this equation instead,

$$\ddot{\theta}+\omega^2\sin\theta=\epsilon e^{i\omega t}. \tag{1.114}$$

1.6 Damped-Driven Chaotic pendulum

In the previous section, we have talked about the damped-driven pendulum with small oscillation. In this section, we will have a brief discussion about the damped-driven pendulum pendulum with large oscillation.

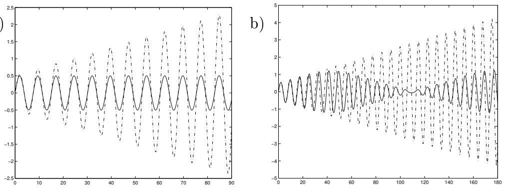


Figure 1.12: The time series of pendulum in resonance for a) small amplitude, b) large amplitude.

The equation we will tackle with is this

$$\ddot{\theta} + \lambda\dot{\theta} + \omega^2 \sin\theta = \epsilon \cos(\Omega t). \tag{1.115}$$

In order to simplify the equation, we have to take away all units from the equation, the technique we use has already discussed in the previous sections, non-dimensionalization.

Observe (1.115), we can find that

$$\begin{aligned} [\lambda] &= t^{-1}, \\ [\omega] &= t^{-1}, \\ [\epsilon] &= t^{-2}, \\ [\Omega] &= t^{-1}. \end{aligned}$$

We choose the unit of time be $\hat{t} = \omega t$, this is to remind that this definition is depended on user, such that you can define the time be $\hat{t} = \Omega t$, and this is also valid. However, in the following, we will base on the first definition. The second step we have to do is equation transformation.

$$\frac{d}{dt} = \omega \frac{d}{d\hat{t}} \tag{1.116}$$

$$\frac{d}{dt^2} = \omega^2 \frac{d^2}{d\hat{t}^2} \tag{1.117}$$

We substite (1.116) and (1.117) into (1.115), so

$$\begin{aligned} \omega^2 \frac{d^2\theta}{d\hat{t}^2} + \lambda\omega \frac{d\theta}{d\hat{t}} + \omega^2 \sin\theta &= \epsilon \cos\left(\frac{\Omega}{\omega}\hat{t}\right) \\ \frac{d^2\theta}{d\hat{t}^2} + \frac{\lambda}{\omega}\frac{d\theta}{d\hat{t}} + \sin\theta &= \frac{\epsilon}{\omega^2}\cos\left(\frac{\Omega}{\omega}\hat{t}\right). \end{aligned} \tag{1.118}$$

(1)	$\frac{\lambda}{\omega} = \frac{1}{q}$		q = damping or quality parameter
(2)	$\frac{\epsilon}{\omega^2} = g$	(not gravity)	g = forcing amplitude
(3)	$\frac{\Omega}{\omega} = \omega_D$		ω_D = driving frequency

For convenience, we will make some changes as tab.1.1. Besides these, we will write t instead of $\hat{t}$ in the following of this section.

Moreover, the three replaced terms are called the non-dimensional group. As we notice in their expression, we can see that they are dimensionless by the cancellation of the numerator and denominator. Therefore, there are three non-dimensional groups. Are there any extra non-dimensional groups? This can be answered base on *Buckingham Pi Theorem*.

Theorem 1 (Buckingham Pi Theorem) *If the equation has N dimensional parameter and M different units, the number of non-dimensional groups is $N - M$.*

Let's examine our equation by this theorem. As we stated before, there has 4 dimension parameter, they are λ, ω, ϵ and Ω; however, only 1 unit in this equation, t. Therefore, the number of non-dimensional groups is 3 according to **Buckingham Pi Theorem** which equals to our statement before.

After the non-dimensionalization and transformation, the new equation is

$$\ddot{\theta} + \frac{1}{q}\dot{\theta} + \sin\theta = g\cos(\omega_D t). \tag{1.119}$$

Decompose this second order ODE into a system of 1^{st} order equations:

$$\begin{aligned} \dot{\theta} &= U \\ \dot{U} &= -\frac{1}{q}U - \sin\theta + g\cos\phi \\ \dot{\phi} &= \omega_D, \end{aligned} \tag{1.120}$$

(1.120) is a system of three 1^{st} order, autonomous equations, and the equations are non-dimensional. The meaning of autonomous is time does not appear explicitly, as we can see θ and ϕ are function of t. Moreover, there has two nonlinear terms in this system, they are $\sin\theta$ and $\cos\phi$.

The necessary conditions for *chaos* are:

1. A system has at least **3** independent dynamical variables,
2. Equations contain a nonlinear coupling.

However, what is *chaos*? The most important characteristic is **sensitive to initial conditions**. That means for a little difference in initial conditions, solution grows exponentially. This is called the **Butterfly Effect**.

Are there any possibility a small oscillation becomes chaos? In fact, linear systems are not sensitive to initial conditions. Say

$$\begin{aligned} \ddot{\theta}_1 + \lambda\dot{\theta}_1 + \omega^2\theta_1 &= \epsilon F(t), \\ \ddot{\theta}_2 + \lambda\dot{\theta}_2 + \omega^2\theta_2 &= \epsilon F(t). \end{aligned}$$

Suppose

$$\delta\theta\left(t=0\right)=\theta_2\left(t=0\right)-\theta_1\left(t=0\right),$$

and

$$\delta\theta=\theta_2-\theta_1.$$

We subtract the two second order ODEs, then

$$\frac{d^2}{dt^2}\left(\delta\theta\right)+\lambda\frac{d}{dt}\left(\delta\theta\right)+\omega^2\delta\theta=0.$$

This is a damped unforced pendulum. Since θ_1 and θ_2 are the solutions for the ODEs, follows from the principle of superposition, their sum and difference is also the solution, therefore $\delta\theta$ is the solution.

Bibliography

[1] J. B. Marsion & S. T. Thornton, *Classical Dynamics of Particles & Systems 3rd edition*, Harcourt Brace Jovan Ovich College Publishers, 1988.