Chapter 2

Tools for studying dynamical systems

2.1 Introduction

In the last chapter, we have covered the basic related to pendulum. Moreover, at the last part we have introduced the concept of chaos. In this chapter, we will further study some interesting behavior of a dynamical system. Parallel to this, we will study the tools for studying dynamical systems, such as *Phase space*, *Poincare section* and *Power spectra*.

2.2 Phase space

The state of motion of a pendulum will be completely specified as a function of time if two quantities are given: the displacement $\theta(t)$ and the velocity $\dot{\theta}(t)$. (Two quantities are needed because the differential equation for the motion is of second order.) We may consider the quantities $\theta(t)$ and $\dot{\theta}(t)$ to be the co-ordinateds of a point in a two-dimensional space, called **phase space**. (In two dimensions, the phase space is a phase plane. But for a general system with n degrees of freedom, the phase soacespace is a $2n$-dimensional space.) As the time varies, the point $P\left(\theta,\dot{\theta}\right)$ describing phase plane. For different initial conditions of the pendulum, the motion will be described by different phase paths. Any given path represents the complete time history of the oscillator for a certain set of initial conditions. The totality of all possible phase paths constitutes the **phase portrait** or the **phase diagram** of the pendulum. Moreover, these considerations are not restricted to pendulum. The concept of phase space is applied extensively in various fields of physics, particularly in statistical mechanics. However, we will mainly use pendulum as an example in this chapter.

2.2.1 Undamped pendulum

Simple Harmonic Oscillation

The equation of simple harmonic oscillation is

$$\ddot{\theta} + \theta = 0. \tag{2.1}$$

Rewrite this into a system of equations

$$\begin{aligned} \dot{\theta} &= U \\ \dot{U} &= -\theta. \end{aligned}$$

The solution of (2.1) is

$$\theta = a_i \cos t + b_i \sin t.$$

If we let the initial conditions be

$$\theta(t=0) = a_i$$

and

$$\dot{\theta}(t=0) = 0.$$

Then,

$$\begin{aligned} \theta &= a_i \cos t \\ \dot{\theta} &= -a_i \sin t \end{aligned}$$

We can calculate the locus of phase path, since

$$\theta^2 + U^2 = a_i^2.$$

Therefore, the locus should be a circular path. Compare fig.2.1 and fig.2.2, we

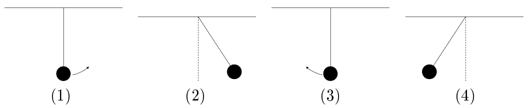

Figure 2.1: The locations of the pendulum corresponding to the position in fig.2.2.

can locate the four locations in fig.2.1 in fig.2.2. Moreover, fig.2.2 shows the locus of the pendulum is a circular orbit. Recall the system of ODE of (2.1), we can write it in a nicer format,

$$\begin{pmatrix} \dot{\theta} \\ \dot{U} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ U \end{pmatrix}. \tag{2.2}$$

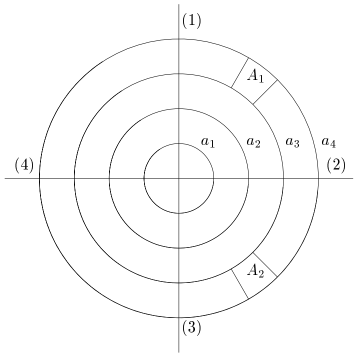

Figure 2.2: The phase spaces of a simple harmonic oscillator with different amplitude of oscillations.

This is a planar system. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.3}$$

$$x = \begin{pmatrix} \theta \\ U \end{pmatrix}, \tag{2.4}$$

(2.2) becomes

$$\dot{x} = Ax. \tag{2.5}$$

Because $Tr(A)(t) = 0$, we find the Wronskian $W(t)$ is preserved,

$$W(t) = W(t_0), \quad \forall \in \mathbf{R}. \tag{2.6}$$

Therefore, the flow in the phase space induced by (2,1) is thus area preserved. As we can see in fig.2.2, $A_1 = A_2$. Moreover, there should be no two phase path can cross for any system. If they could cross, this would imply that for a given set of initial conditions, the motion could proceed along different phase paths. But this is impossible since the solution of the differential equation is unique.

Large oscillation

The system of equation of large oscillation is

$$\dot{\theta} = U$$

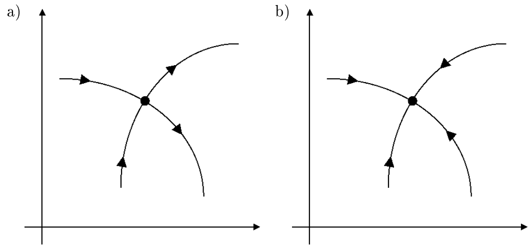

Figure 2.3: These two figures show two situation of paths cross over, a) is incorrect and b) is correct.

$$\dot{U} = -\sin\theta. \tag{2.7}$$

Due to the relationship of $\sin\theta$ and θ, for the small value of initial value, the loci should be circular. However, will it still be circular for large oscillation? The loci of large oscillation is shown in fig.2.4. We can see that the circular locus no longer exist for large oscillation but replaced by a ellipse like orbit. However, this also does not last for every case, as the amplitude increase continuously, the orbit transforms to the shape neither circular nor elliptic. While it looks like an eye.

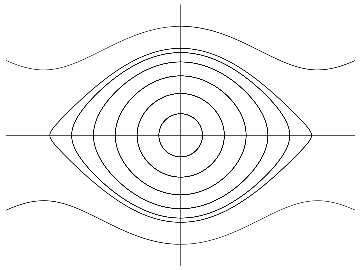

Figure 2.4: Phase space of large oscillaion pendulum without damping and external forces.

$\lambda^2 - 4\omega^2 < 0$	$\Rightarrow$	$\frac{1}{q^2} < 4$	*underdamping*
$\lambda^2 - 4\omega^2 = 0$	$\Rightarrow$	$\frac{1}{q^2} = 4$	*critical-damping*
$\lambda^2 - 4\omega^2 > 0$	$\Rightarrow$	$\frac{1}{q^2} > 4$	*overdamping*

Table 2.1: Relationship between the value of parameters and the type of dampings.

2.2.2 Unforced damped pendulum

We just discussed the phase space of a simple harmonic oscillator, how about the case for those damped pendulum? Recall the general equation for unforced damped pendulum,

$$\ddot{\theta} + \frac{1}{q}\dot{\theta} + \theta = 0, \tag{2.8}$$

$$\begin{pmatrix} \dot{\theta} \\ \dot{U} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{1}{q} \end{pmatrix} \begin{pmatrix} \theta \\ U \end{pmatrix}. \tag{2.9}$$

Its auxiliary equation is

$$\alpha^2 + \frac{1}{q}\alpha + 1 = 0, \tag{2.10}$$

since $\frac{1}{q} = \frac{\lambda}{\omega}$, therefore, we can summarize the cases for damping as table.2.1.

Here, (2.8) is the general equation while (2.9) is its planar system form. From (2.8), we can see that the trace of the matrix not equals to zero, therefore, we can have a idea that the area of phase space will not be conserved any more. This would be true for every one of the three damped pendulum.

Underdamped pendulum

In the case of underdamped pendulum, the solution of the auxiliary equation are

$$\alpha_{\pm} = \frac{-\frac{1}{q} \pm \sqrt{\frac{1}{q^2} - 4}}{2}. \tag{2.11}$$

Since it is underdamped pendulum, the value of determinant should less than zero. We let the followings

$$\gamma_1 = \frac{1}{2q} \quad \gamma_2 = \frac{1}{2}\sqrt{4 - \frac{1}{q^2}},$$

then the solution of θ and its first derivative are

$$\theta = Ae^{-\gamma_1 t}\cos(\gamma_2 t - \delta), \tag{2.12}$$

$$\dot{\theta} = -Ae^{-\gamma_1 t}\left[\gamma_1 \cos(\gamma_2 t - \delta) + \gamma_2 \sin(\gamma_2 t - \delta)\right]. \tag{2.13}$$

We further let these ,

$$x = \gamma_2 \theta, \quad y = \gamma_1 \theta + \dot{\theta}.$$

So,

$$x = \gamma_2 Ae^{-\gamma_1 t}\cos(\gamma_2 t - \delta),$$
$$y = -\gamma_2 Ae^{-\gamma_1 t}\sin(\gamma_2 t - \delta).$$

We, then, change them from rectangular coordinate (x, y) to polar coordinate (r, ζ),

$$r = \sqrt{x^2 + y^2}, \quad \zeta = \gamma_2 t,$$

$$r = \gamma_2 Ae^{-\left(\frac{\gamma_1}{\gamma_2}\right)\zeta}. \tag{2.14}$$

Since γ_1 and γ_2 are positive, their division should be positive also. From (2.14), we can know that the locus of underdamped pendulum is spiral like which is shown in fig.2.5.a. Observe fig.2.5.a carefully, the distance between two paths is

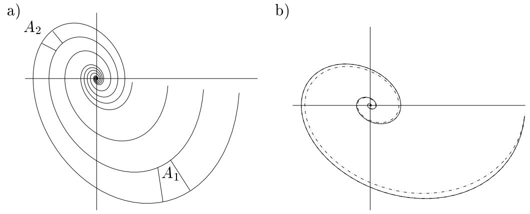

Figure 2.5: Phase space of underdamped pendulum for a) small oscillation, b) large oscillation.

decreasing. If we pick up two area as last section, we found that $A(t_1) > A(t_2)$. This shows that a underdamped pendulum is a dissipative system. The locus for large oscillation is also a spiral like, however, it seems to be compressed (fig.2.5.b).

Overdamped pendulum

Since the determinant is greater than zero in the case of overdamped pendulum, the solution of θ can be written as,

$$\theta = Ae^{\gamma_+ t} + Be^{\gamma_- t}, \tag{2.15}$$

where

$$\gamma_{\pm} = -\frac{1}{2q} \pm \frac{1}{2}\sqrt{\frac{1}{q^2} - 4}.$$

Therefore, the first derivative of θ is

$$\dot{\theta} = A\gamma_+ e^{\gamma_+ t} + B\gamma_- e^{\gamma_- t}. \tag{2.16}$$

For large time t, the first term (2.15) and (2.16) will get to zero much faster than the second one, such that

$$\begin{aligned} \theta &= Be^{\gamma_- t}, \\ \dot{\theta} &= B\gamma_- e^{\gamma_- t}, \\ \ddot{\theta} &= \gamma_- \theta. \end{aligned} \tag{2.17}$$

This is the slope-intercept form of a line with y-intercept at zero and slope γ_- which is independent of the initial value and time. Therefore, all the curves have different initial values, but end at the origin (fig.2.6.a).

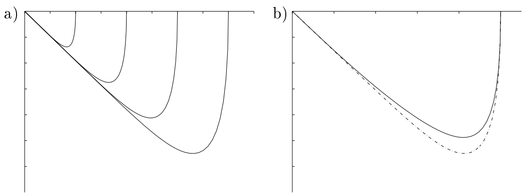

Figure 2.6: Phase space of overdamped pendulum with a) small oscillation, b) large oscillation.

In fig.2.6.a, the loci of different phase paths seems to be a straight line some time after the starting. However, this is just a illusion, in fact, the lines do not merge together, but just near to others (fig.2.7). Why they can not be the same line? This is because if they can be the same line, that means some time later the solutions will be the same no matter what initial values are. Therefore, there is more than one solution for a particular initial value problem which is contradicted to the uniqueness of the solution.

Critical-damped pendulum

The last case of damping have not discussed is critical-damped pendulum. Since the determinant of auxiliary equation equals to zero, we can have one solution directly only. However, if we apply the reduction of order, the second solution can be found and the general solution of this case is

$$\theta = (A + Bt)\, e^{-gammat}. \tag{2.18}$$

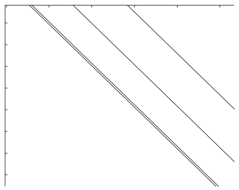

Figure 2.7: The zoom in of the "colinear" lines.

The corresponding first derivative is

$$\dot{\theta} = -\gamma\,(A + Bt)\, e^{-\gamma t} + Be^{-\gamma t}. \tag{2.19}$$

We can have the relationship between θ and $\dot{\theta}$, such that

$$\dot{\theta} = -\gamma\theta + Be^{-\gamma t}. \tag{2.20}$$

This is a slope-intercept form. This time, y-intercept is a expression of $Be^{-\gamma t}$, which not equals to zero. However, since γ is positive, therefore, as t increase, $e^{-\gamma t}$ will decrease and tends to zero. As a result, for larger value of t, y-intercept still equals to zero. Base on this, we can draw a conclusion that, the phase spaces of critical-damped pendulum look like the phase spaces of overdamped pendulum (fig.2.8).

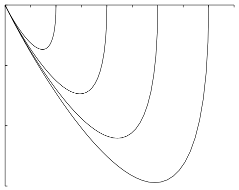

Figure 2.8: Phase space of critical-damped pendulum.

Attractors and Basins of Attraction

As we can see in these three cases, underdamped, overdamped and critical-damped pendulum, their phase paths must approach the point of origin, such that

$$(\theta, U) \quad \Rightarrow \quad (0, 0).$$

The origin is called an attractor or point of attraction. An attractor is a stable configuration in phase space. However, there should be more than one point of attraction, due to the exisitence of basins of attraction (fig2.9).

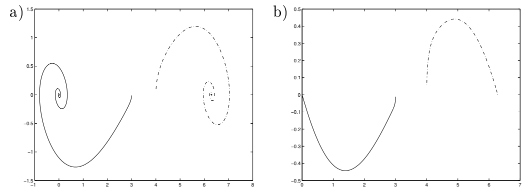

Figure 2.9: These two figures show there are more than one attractors for the motion of pendulum, a) underdamped pendulum and b) overdamped pendulum.

A domain of attraction of an asymptotically stable stationary point x_0 of the system $\dot{x} = f(x)$ consists of points for which solutions starting at these points approach x_0 for $t \to \infty$. Among these domains, the maximum domain of attraction is called the basin of attraction. Up to now, the stationary point we have is the origin, however, in different basin of attraction, they should have their stationary point.

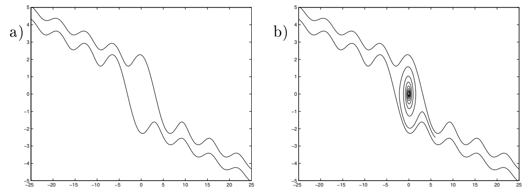

Figure 2.10: These figures show one of the basins of attraction for the case of underdamped pendulum. Moreover, b) shows us that for arbitiary initial values, if they are within this basins of attraction, the same attractor with be the same.

2.3 Poincare Section

A Poincare section is a device invented by Henri Poincare as a means of simplifying phase space diagrams of complicated systems. It is constructed by viewing the phase space diagram stroboscopically in such a way that the motion is observed periodically. For the driven pendulum, the strobe period is the period of the forcing, $\frac{1}{\omega_D}$.

$$\begin{aligned} \dot{\theta} &= U \\ \dot{U} &= -\frac{1}{q}U - \sin\theta + g\cos\phi \\ \dot{\phi} &= \omega_D \end{aligned} \tag{2.21}$$

Therefore, instead of reporting $(\theta,\ \omega)$ in arbitrary interval, the interval is chosen to be every $\frac{2\pi}{\omega_D}$.

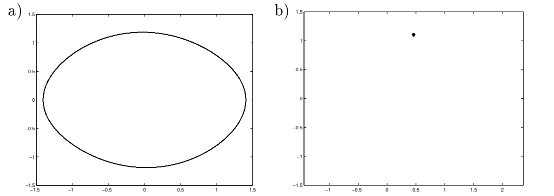

Figure 2.11: a) Phase space and b) poincare section of period 1 motion.

However, not all system has the forced frequency as pendulum, such as **Lorenz** Equations and **Chua's Circult** Equations. Therefore, another method should be used. Instead of using the forced frequency, we use the device called poincare plane, it can be visualized as fig.2.13.

The idea of poincare plane is that: suppose there is a phase path in N dimension, and a plane with $N-1$ dimension is placed acrossed the path. Therefore, there should be some intercepting points on the plane, such intercepting points will plot the diagram which we called poincare section. The position of the poincare plane can be put anywhere, the main concern is does there any path passing through the plane. From this we know that in the equation is N dimension, then the poincare section should be $N-1$ dimension.

For a dynamical system, the Poincare section provides a simplification of the phase diagram while retaining the essential features of the dynamics. Moreover for the periodic motions, the appearance of these sections is quite simple. However, where the chaotic behavior is described, the simplification of phase space

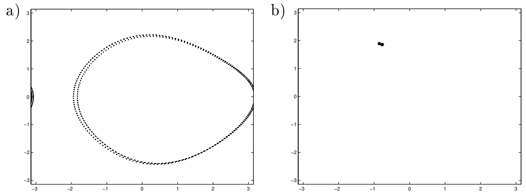

Figure 2.12: a) Phase space and b) poincare section of period 2 motion.

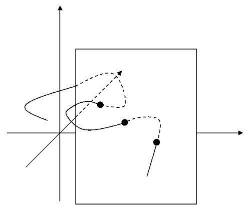

Figure 2.13: The visualization of the idea of poincare plane.

provided is shown to be very important for an understanding of the physics. One useful tool for the study of chaos is the observation of the distribution o points on a computer-generated Poincare section.

2.4 Spectral Analysis Of Time Series

2.4.1 Fourier Series For Periodic Functions

Assume $f(t)$ is periodic with period T, i.e. for any integer n

$$f(t+nT) = f(t)\,, \tag{2.22}$$

for all t. Therefore, by using Fourier expansion, we have

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\omega_n t + b_n \sin\omega_n t\right)\,, \tag{2.23}$$

with $\omega_n = n\tilde{\omega}$ and $\tilde{\omega} = 2\pi/T$.

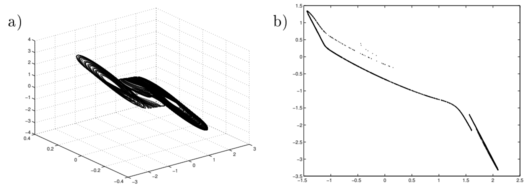

Figure 2.14: a) The phase diagram of Chua's Circult equation and b) it's corresponding poincare section.

Orthagonality Of Sine and Cosine

If we let $\delta_{m,n}$ be the Kronecker delta meaning

$$\delta_{m,n} = \begin{cases} 1 & , \quad m = n \\ 0 & , \quad m \neq n\,, \end{cases} \tag{2.24}$$

we have the following orthogonal relations

$$\begin{aligned}
\int_0^T \sin(\omega_m t)\sin(\omega_n t)dt &= \begin{cases} \frac{1}{2}T\delta_{m,n} & , \quad m\neq 0 \\ 0 & , \quad m = 0\,, \end{cases} \\
\int_0^T \cos(\omega_m t)\cos(\omega_n t)dt &= \begin{cases} \frac{1}{2}T\delta_{m,n} & , \quad m\neq 0 \\ T & , \quad m = 0\,, \end{cases} \\
\int_0^T \sin(\omega_m t)\cos(\omega_n t)dt &= 0\,.
\end{aligned} \tag{2.25}$$

By using these identities, for any periodic function, we have

$$\begin{aligned}
\int_0^T f(t)\cos\omega_m t dt &= \frac{1}{2}a_0\int_0^T \cos\omega_m t dt + \sum_{n=1}^{\infty} a_n \int_0^T \cos\omega_m t\cos\omega_n t dt \\
&\quad + b_n \int_0^T \sin\omega_n t \cos\omega_m t dt \\
&= \frac{1}{2}a_m T\,,
\end{aligned} \tag{2.26}$$

and therefore

$$a_m = \frac{2}{T}\int_0^T f(t)\cos\omega_m t dt\,, \tag{2.27}$$

for $m\neq 0$. If $m = 0$, we can consider

$$\int_0^T f(t)dt = \frac{1}{2}a_0\int_0^T dt = \frac{T}{2}a_0\,, \tag{2.28}$$

which gives

$$a_0 = \frac{2}{T}\int_0^T f(t)\cos\omega_0 t dt\,. \tag{2.29}$$

By using similar method, b_m can be computed as

$$b_m = \frac{2}{T}\int_0^T f(t)\sin\omega_m t dt\,, \tag{2.30}$$

for $m = 1, 2, \cdots$.

As a result, with $i = \sqrt{-1}$, the periodic function $f(t)$ can be written as

$$\begin{aligned} f(t) &= \frac{1}{2}a_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n\left(e^{i\omega_n t} + e^{-i\omega_n t}\right) - ib_n\left(e^{i\omega_n t} + e^{-i\omega_n t}\right)\right] \\ &= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{2}(a_n - ib_n)e^{i\omega_n t} + \sum_{n=-\infty}^{-1}\frac{1}{2}(a_{-n} + ib_{-n})e^{i\omega_n t} \\ &= \sum_{n=-\infty}^{\infty}\hat{f}_n e^{i\omega_n t}\,, \end{aligned} \tag{2.31}$$

with

$$\hat{f}_0 = \frac{1}{2}a_0\,,\ \hat{f}_n = \frac{1}{2}(a_i - ib_n) \text{ and } \hat{f_{-n}} = \frac{1}{2}(a_n + ib_n)\,, \tag{2.32}$$

for $n \geq 1$. Therefore

$$\hat{f_{-n}} = \hat{f_n}^*\,. \tag{2.33}$$

By using the orthogonality

$$\begin{aligned} \int_0^T e^{i(\omega_m - \omega_n)t}dt &= \int_0^T \left[\cos(\omega_m - \omega_n)t + i\sin(\omega_m - \omega_n)t\right]dt \\ &= T\delta_{m,n}\,, \end{aligned} \tag{2.34}$$

$\hat{f_m}$ can be calculated by

$$\hat{f_m} = \frac{1}{T}\int_0^T f(t)e^{-i\omega_m t}dt\,. \tag{2.35}$$

Now, consider the coefficients again. By writing

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(t)\cos\omega_n t dt = \frac{2}{T}\int_{-\pi/2}^{\pi/2} f(t)\cos\omega_n t dt \\ b_n &= \frac{2}{T}\int_0^T f(t)\sin\omega_n t dt = \frac{2}{T}\int_{-\pi/2}^{\pi/2} f(t)\sin\omega_n t dt\,, \end{aligned} \tag{2.36}$$

b_n would be equal to zero if the periodic function $f(t)$ is an even function, and a_n would equal zero if $f(t)$ is an odd function.

Example: Sawtooth Wave.

Consider a sawtooth wave given by

$$f(t) = \begin{cases} t & , \quad x \leq t \leq \pi \\ t - 2\pi & , \quad \pi < t \leq 2\pi. \end{cases} \tag{2.37}$$

Then, by applying the above discrete Fourier transform to this periodic function with period 2π, we have

$$f(t) = \sum_{n=-\infty}^{\infty}\hat{f}_n e^{i\omega_n t}\,, \tag{2.38}$$

with

$$\begin{aligned} \hat{f}_n &= \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-i\omega_n t}dt \\ &= \frac{1}{2\pi}\int_0^{\pi} te^{-i\omega_n t}dt + \frac{1}{2}\int_{\pi}^{2\pi}(t - 2\pi)e^{-i\omega_n t}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{-i\omega_n t}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{-int}dt \\ &= \frac{i}{n}(-1)^n\,, \end{aligned} \tag{2.39}$$

for $n \neq 0$, and

$$\hat{f}_0 = 0\,. \tag{2.40}$$

As a result,

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}2\sin nt \\ &= 2\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \cdots\right)\,. \end{aligned} \tag{2.41}$$

2.4.2 Fourier Series For Non-periodic Functions

If the function $f(t)$ is not periodic, then the Fourier series would be replaced by a Fourier integral. For a periodic function, we have

$$f(t) = \sum_{-\infty}^{\infty}\hat{f}_n e^{i\omega_n t} \text{ and } \hat{f}_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-i\omega_n t}dt\,, \tag{2.42}$$

with $\omega_n = n\tilde{\omega}$ and $\tilde{\omega} = 2\pi/T$. If the function is not periodic, which means $T \to \infty$, i.e. $\tilde{\omega} \to 0$. Therefore, we have

$$f(t) = \sum_{n=-\infty}^{\infty}\frac{\hat{f}_n}{\tilde{\omega}\Delta n}e^{i\omega_n t}(\tilde{\omega}\Delta n)\,. \tag{2.43}$$

by letting

$$\Delta n = 1\,,\ \omega = \omega_n = \tilde{\omega} n\,,\ d\omega = \tilde{\omega}\Delta n \ \text{ and } \ \hat{f}(\omega) = \frac{\hat{f}_n}{\tilde{\omega}\Delta n}\,, \tag{2.44}$$

we have

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega \\ \hat{f}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt\,. \end{aligned} \tag{2.45}$$

2.4.3 Dirac Delta Function

The Dirac delta function is defined as

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} d\omega\,. \tag{2.46}$$

If we consider the Fourier integral again, we have

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s) e^{-i\omega s} ds\right] e^{i\omega t} d\omega \\ &= \int_{-\infty}^{\infty} f(s)\delta(s-t) ds\,. \end{aligned} \tag{2.47}$$

One interesting property of this delta function is that when $f(x) = 1$, we have, from the definition,

$$\int_{-\infty}^{\infty} \delta(x-a) dx = 1\,, \tag{2.48}$$

for all a.

Example 1.

Instead of integrating from negative infinity to infinity, we define

$$\delta_n(x) = \frac{1}{2\pi}\int_{-n}^{n} e^{ixt} dt\,. \tag{2.49}$$

Therefore, by simplifying the expression, we obtain

$$\delta_n(x) = \frac{1}{\pi}\frac{\sin n\pi}{x}\,. \tag{2.50}$$

Fig.2.15 shows the shape of this "delta function". If we consider the are under this curve, we find

$$\int_{-\infty}^{\infty} \delta_n(x) dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin nx}{x} dx = 1\,. \tag{2.51}$$