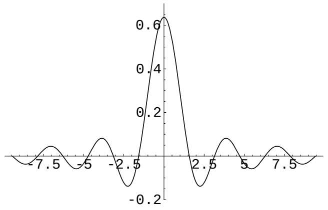

Figure 2.15: Delta function with parameter $n = 2$.

2.4.4 Power Spectrum

Now, we define the spectrum function $S(\omega)$ as

$$S(\omega) = \hat{f}(\omega)\hat{f}(\omega)^* = \left|\hat{f}(\omega)\right|^2 . \tag{2.52}$$

Then, from the Parseval's Theorem, if $f(t)$ is a real function, we have

$$\begin{aligned}\int_{-\infty}^{\infty} S(\omega)d\omega &= \hat{f}(\omega)\hat{f}(\omega)^* d\omega \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right]\left[\int_{-\infty}^{\infty} f(t')e^{i\omega t'}dt'\right]d\omega \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' f(t)f(t')\delta(t-t') \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt f(t)f(t) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}[f(t)]^2\, dt .\end{aligned} \tag{2.53}$$

Example 1.

Consider the function

$$f(t) = \begin{cases} e^{-\gamma t}e^{i\omega_0 t} & , \quad t \geq 0 \\ 0 & , \quad t < 0 . \end{cases} \tag{2.54}$$

From this, we have

$$\hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

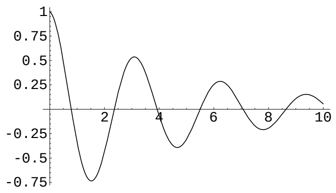

Figure 2.16: Shape of the function in example 1 with parameter $\gamma = 0.2$ and $\omega_0 = 2.0$.

$$\begin{aligned} &= \frac{1}{2\pi}\int_0^{\infty} e^{-\gamma t}e^{i\omega_0 t}e^{-i\omega t}dt \\ &= \frac{1}{2\pi}\int_0^{\infty} e^{-t[\gamma+i(\omega-\omega_0)]}dt \\ &= \frac{1}{2\pi[\gamma+i(\omega-\omega_0)]} .\end{aligned} \tag{2.55}$$

Therefore

$$\begin{aligned} S(\omega) &= \frac{1}{(2\pi)^2[\gamma+i(\omega-\omega_0)][\gamma-i(\omega-\omega_0)]} \\ &= \frac{1}{(2\pi)^2}\frac{1}{[\gamma^2+(\omega-\omega_0)^2]} .\end{aligned} \tag{2.56}$$

2.4.5 Spectral Analysis

The purpose of the spectrum is that it can be used as an indicator of the appearance of chaos. Broadband power spectrum is an indictive of chaos. Suppose we have some signal $\theta(t)$ and we want to find out it's power spectrum, $|\theta(\omega)|^2$.

For example, if we have the discrete values of θ, i.e.

$$\theta_k = \theta(t_k) , \; t_k = k\Delta t , \tag{2.57}$$

with $k = 0, 1, \cdots, N-1$, we first find out the values of $\hat{\theta}(f_n)$ at

$$f_n = \frac{n}{N\Delta t} , \; n = \frac{-N}{2}+1, \cdots, \frac{N}{2} . \tag{2.58}$$

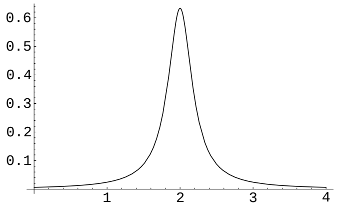

Figure 2.17: Spectrum function $S(\omega)$ of the function in example 1 with $\gamma = 0.2$ and $\omega_0 = 2.0$.

Because

$$\hat{\theta}(f_n) = \int_{-infty}^{\infty} \theta(t) e^{i2\pi f_n t} dt \,, \tag{2.59}$$

we approximate it as

$$\begin{aligned} \hat{\theta}(f_n) &\simeq \sum_{k=0}^{N-1} \theta_k e^{i2\pi f_n t_k} \Delta t \\ &= \Delta t \sum_{k=0}^{N-1} \theta_k e^{i2\pi nk/N} \,. \end{aligned} \tag{2.60}$$

Now, we define

$$\hat{\theta_n} = \sum_{k=0}^{N-1} \theta_k e^{i2\pi nk/N} \tag{2.61}$$

as the discrete Fourier transform of the N data points of θ_k. From this, we see that if $\theta(t)$ is real,

$$\hat{\theta}_{-n} = \hat{\theta}_n^* \,. \tag{2.62}$$

Since $\hat{\theta_n}$ is periodic in the variable n with period N, we can write

$$\hat{\theta}_{-n} = \hat{\theta}_{N-n} \text{ or } \hat{\theta}_{N/2+1} = \hat{\theta}_{-N/2+1} \,. \tag{2.63}$$

Therefore, instead of using $n = -N/2+1, \cdots, N/2$, we use $n = 0, 1, \cdots, N-1$ for usual convention.

Now, we define the Power Spectrum for $f > 0$ as

$$P(f_n) = \frac{1}{N^2} \left| \hat{\theta}_n \right|^2 \,, \tag{2.64}$$

with $f_n = n/(N\Delta t)$. For example,

$$P(0) = \frac{1}{N^2} \left| \hat{\theta}_0 \right|^2 = \frac{1}{N^2} \left| \sum_{k=0}^{N-1} \theta_k \right|^2 = \left| \frac{1}{N} \sum_{k=0}^{N-1} \theta_k \right|^2 = < \theta >^2 \,. \tag{2.65}$$

2.4.6 Numerical Method

When we are using the fft package in the MATLAB, several point should be noticed.

1. N should be in a form of
$$N = 2^p \,, \tag{2.66}$$
for some integer p. Normally, p would be chosen to be 18, such that $N = 262,144$.
2. N and Δt are chosen in a way that $N\Delta t$ should include several oscillation of the motion.
3. Instead of using θ directly, it would be better to consider $\dot{\theta}(t_k)$. The reason is that θ may be unbounded as θ is a periodic variable.

Coding

The MATLAB code can be written as

```
N=2∧18;
delta=(Δ t of the time series);
uf=fft(u)/N;
p=uf.*conj(uf);
t=[0:N-1]*delta;
f=[0:N/2]/(N*delta);
semilogy(f,p(1:N/2+1));
axis([0 1 10∧-6 1]) ;
```

Example

Consider the function

$$u(t) = \sin(2\pi t) + \frac{1}{4}\sin(\frac{\pi}{8}t) \,. \tag{2.67}$$

Therefore, from the analytical studies, we found that

$$P(f) = \begin{cases} 0.25 & , \quad f = 1 \\ 0.125625 & , \quad f = 0.0625 \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{2.68}$$

Figure (??a) shows the plot of the function $u(t)$ versus t and figure (??b) shows the power spectrum function $P(f)$ of this function $u(t)$.

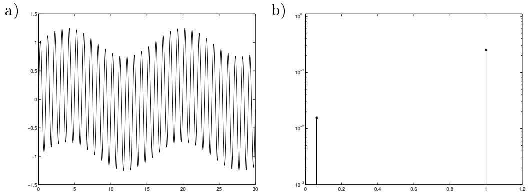

Figure 2.18: (a) $u(t)$ versus t, and (b) $P(f)$ versus f.

2.5 Visualization of the pendulum's dynamics

Up to now, we have discussed some tools studying the pendulum's dynamics. In this section, we will have some visualization of the chaotic behavior of pendulum motion.

2.5.1 Sensitivity to initial conditions

As we stated in the last chapter, a feature of chaotic system is its sensitivity to initial conditions. For a small changes of initial conditions, the solution will have exponentially difference. Such behavior can be easily seen in pendulum system as fig.2.19.

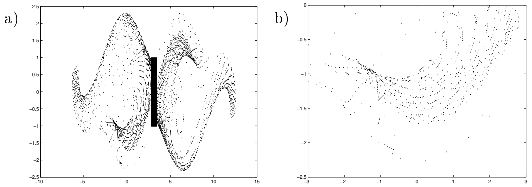

Figure 2.19: These show the situation of sensitive to initial conditions. The black block in a) are the initial values. b) is the zoom in of some region in a).

2.5.2 Poincare section

In the previous section, we have introduced the concept of poincare section and two poincare sections are shown which are period 1 and period 2. These seem not so interesting. For the case of chaotic system, the appearance of the poincare section is totally different from these, such phase space and the corresponding poincare section are shown in fig.2.20 and fig.2.21 respectivitily.

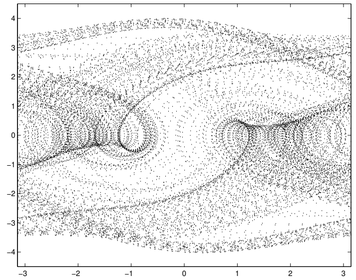

Figure 2.20: Phase space of the chaotic motion of pendulum.

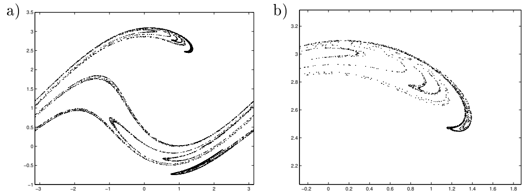

Figure 2.21: The poincare section of fig.2.20, b) is the zoom in of this.

In observe fig.2.16 and its magnification carefully, we should find that there has some self similar structure. This is caused by the folding and stretching of phase volume. This self repeating structure is called fractal, and there has a number called fractal dimension related to this. These concepts will be discussed in the next section.

2.5.3 Bifurcation diagrams

In the section related to poincare section, we find that for differen value of the parameters, the period of the system is changed. However, phase diagrams, poincare sections, time series and power spectra can only provide information about the dynamics of the system for specific values of the parameters. The dynamics may also be viewed more globally over a range of parameter values, thereby allowing simultaneous comparison of periodic and chaotic behavior. The bifurcation diagram provides a summary of the essential dynamics and is therefor a useful method of acquiring this overview.

However, what is bifurcation. Use pendulum as example: for some values of the parameters, a pendulum will have only one long-term motion, while for other slightly different choices, two or more motions may be possible. If several of them are stable, the actual number of solutions to a differential equation as a parameter is varies is called a bifurcation.

Period Doubling Phenomena

We use Logistic map as an example to illustrate the phenomena of period doubling. Logistic map is an one dimensional map and its expression is

$$x_{n+1} = f_\mu(x_n), \tag{2.69}$$

μ is the parameter and it is the only parameter of this equation, and x_n is the n^{th} value of x. The task is for any given initial value x_o, we have to find out x_1, x_2, $\cdots$. Therefore, for different value of μ, the sequence should not be the same. The iteration process is

$$\begin{aligned}
x_1 &= f_\mu(x_0) \\
x_2 &= f_\mu(x_1) = f_\mu(f_\mu(x_0)) \\
x_3 &= f_\mu(x_1) = f_\mu(f_\mu(f_\mu(x_0)) = f_\mu^{(3)}(x_0) \\
&\vdots \\
x_{n+1} &= f_\mu(x_n) = f_\mu^{(n+1)}(x_0).
\end{aligned}$$

There are some modified Logistic map, however, the one we will study is the simpliest one. The linear case is

$$x_{n+1} = \mu x_0. \tag{2.70}$$

Therefore, the iteration process becomes

$$\begin{aligned}
x_1 &= \mu x_0 \\
x_2 &= \mu^2 x_0 \\
&\vdots \\
x_{n+1} &= \mu^{n+1} x_0.
\end{aligned}$$

For this case, there has only two possibilities,

$$\begin{aligned}
|\mu| < 1 \quad &\Rightarrow \quad x_n \to 0 \\
|\mu| > 1 \quad &\Rightarrow \quad x_n \to \infty
\end{aligned}$$

as $n \to \infty$. This is a trivial dynamics without any interesting thing. We are interested in its non-linear case

$$x_{n+1} = \mu x_n(1 - x_n) = f_\mu(x_n). \tag{2.71}$$

We first assign the initial value, x_0, such that $0 \le x_0 \le 1$; and then, same as the linear case, iterate this many times.

$$\begin{aligned}
x_1 &= \mu x_0(1 - x_0) && \text{(polynomial of order 2 in } x_0\text{)} \\
x_2 &= \mu x_1(1 - x_1) && \text{(polynomial of order 4 in } x_0\text{)} \\
&= \mu(\mu x_0(1 - x_0))(1 - \mu x_0(1 - x_0)) \\
x_3 &= \mu x_2(1 - x_2) && \text{(polynomial of order 8 in } x_0\text{)} \\
&\vdots \\
x_n &= \mu x_{n-1}(1 - x_{n-1}) && \text{(polynomial of order } 2^n \text{ in } x_0\text{)}
\end{aligned}$$

The maximum of (2.71) is

$$\begin{aligned}
f_\mu'(x_m) = 0 \;\Rightarrow\; \mu - 2\mu x_m &= 0 \\
x_m &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
f_\mu\left(\frac{1}{2}\right) &= \frac{1}{2}\mu\left(1 - \frac{1}{2}\right) \\
&= \frac{\mu}{4}.
\end{aligned}$$

Fig.2.12 shows the situation of the Logistic map with μ equals to 2. We can see the sequence of convergence from the arbitrary initial value to the fixed point, $x = \frac{1}{2}$. If we compare this with the maximum, it agrees with the expression $\frac{\mu}{4}$.

However, the fixed point is varied by the parameter μ. We have to find the fixed point x^* as expression of μ, such that $x^* = f_\mu(x^*)$.

$$\begin{aligned}
x^* &= f_\mu(x^*) \\
&= \mu x^*(1 - x^*), \\
x^*(\mu(1 - x^*) - x^*) &= 0.
\end{aligned} \tag{2.72}$$

$x^* = 0$ is one of the solution for (2.25), another one is

$$\begin{aligned}
\mu(1 - x^*) - x^* &= 0 \\
\mu x^* &= \mu - 1 \\
x^* &= 1 - \frac{1}{\mu}.
\end{aligned} \tag{2.73}$$

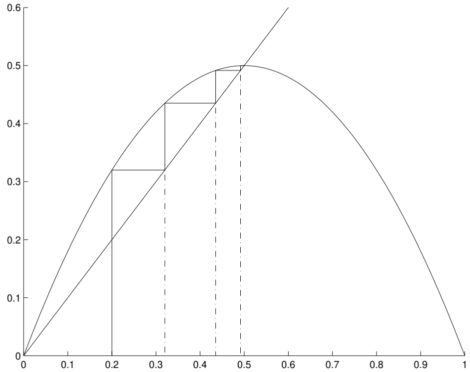

Figure 2.22: This shows the situation of convergence of the initial value of x.

Since $0 \leq x_0 \leq 1$, therefore for $0 \leq \mu < 1$, only $x^* = 0$ is a fixed point; and for $\mu > 1$, both 0 and $1 - \frac{1}{\mu}$ are fixed points. Is the fixed point stable or unstable (attracting or repelling)?
Let x^* be a fixed point such that $x^* = f(x^*)$ and $x_n = x^* + \epsilon_n$, $\epsilon \ll 1$,

$$\begin{aligned} x_{n+1} &= f(x_n) \\ &= f(x^* + \epsilon_n) \\ &= f(x^*) + \epsilon f'(x^*) + \text{higher order in } \epsilon_n \\ &\simeq x^* + \epsilon_n f'(x^*) \,. \end{aligned}$$

by definition $x_{n+1} - x^* = \epsilon_{n+1}$

$$\begin{aligned} \frac{\epsilon_{n+1}}{\epsilon_n} &= f'(x^*)\,, \\ \left|\frac{\epsilon_{n+1}}{\epsilon_n}\right| &= |f'(x^*)|\,. \end{aligned} \tag{2.74}$$

If $|f'(x^*)| \begin{cases} < 1 & x^* \text{ is stable} \\ > 1 & x^* \text{ is unstable} \\ = 1 & \text{marginally stable} \\ = 0 & x^* \text{ is superstable} \end{cases}$. The fixed points of $f_\mu(x)$ are

$$x^* = 0, \qquad x^* = 1 - \frac{1}{\mu}.$$

$$f'_\mu(x) = \mu - 2\mu x = \mu(1 - 2x) \tag{2.75}$$

Therefore

$$f'_\mu(0) = \mu, \qquad \text{stable for } 0 \leq \mu < 1. \tag{2.76}$$

$$\begin{aligned} f'_\mu\left(1 - \frac{1}{\mu}\right) &= \mu\left(1 - 2\left(1 - \frac{1}{\mu}\right)\right) \\ &= \mu\left(-1 + \frac{2}{\mu}\right) \\ &= 2 - \mu, \end{aligned} \tag{2.77}$$

so $\left|f'_\mu\left(1 - \frac{1}{\mu}\right)\right| < 1$ for stability. This implies

$$\begin{aligned} &|2 - \mu| < 1 \\ &1 < \mu < 3. \end{aligned} \tag{2.78}$$

Therefore, for period one

$$\begin{aligned} 0 \leq \mu \leq 1 \qquad & x_n \to 0 \\ 0 \leq \mu \leq 3 \qquad & x_n \to 1 - \frac{1}{\mu} \end{aligned}$$

as $n \to \infty$.

What will happens if $\mu > 3$? This will look for a period two cycle. We consider the fixed point of the following mapping

$$x_{n+1} = f_\mu(f_\mu(x_n))\,. \tag{2.79}$$

Therefore, the fixed points are

$$\begin{aligned} x^* &= f_\mu(f_\mu(x^*)) \\ &= g_\mu(x^*)\,. \end{aligned} \tag{2.80}$$

The stability of period 2-cycle is $\left|g'_\mu(x^*)\right| < 1$. Since we are looking for the fixed points of period 2-cycle, that means it should has two fixed points, we let them be x_1^* and x_2^* respectively,

$$x_1^* = f_\mu(x_2^*)\,, \tag{2.81}$$

$$x_2^* = f_\mu(x_1^*)\,. \tag{2.82}$$

Then,

$$\begin{aligned} \left|g'_\mu(x_1^*)\right| &= \left|f'_\mu(f_\mu(x_1^*))\, f'_\mu(x_1^*)\right| \\ &= \left|f'_\mu(x_2^*)\, f'_\mu(x_1^*)\right|. \\ \left|g'_\mu(x_2^*)\right| &= \left|f'_\mu(f_\mu(x_2^*))\, f'_\mu(x_2^*)\right| \\ &= \left|f'_\mu(x_1^*)\, f'_\mu(x_2^*)\right|. \end{aligned}$$

As a result, both fixed points of $g_\mu(x)$ become unstable at same μ.

Now, we let the fixed point of period 2-cycle be x^* again, so $x^* = f_\mu\left(f_\mu\left(x^*\right)\right)$.

$$\begin{aligned} x^* &= f_\mu\left(f_\mu\left(x^*\right)\right) \\ &= f_\mu\left(\mu x^*\left(1-x^*\right)\right) \\ &= \mu\left(\mu x^* \mu\left(1-x^*\right)\left(1-\mu x^*\left(1-x^*\right)\right)\right) \\ &= \mu^2 x^*\left(1-x^*\right) - \mu^3 x^{*2}\left(1-x^{*2}\right) \end{aligned} \tag{2.83}$$

From the previous works, we know 2 solutions already, they are

$$x^* = 0, \qquad x^* = 1 - \frac{1}{\mu}.$$

Therefore, we can factor out these two from (2.83), then it becomes

$$\mu^3 x^{*2} - \left(\mu^3 + \mu^2\right) x^* + \left(\mu^2 + \mu\right) = 0. \tag{2.84}$$

This is a quadratic equation in x^*, and it solution are

$$x^*_\pm = \frac{1}{2\mu}\left[(\mu+1) \pm \sqrt{(\mu+1)(\mu-3)}\right]. \tag{2.85}$$

This is still period 1 for $\mu < 3$, so it will become period 2-cycle when $\mu = 3$. Therefore

$$x^*_\pm = \frac{2}{3} \qquad (1 - \frac{1}{\mu}|_{\mu=3} = \frac{2}{3}). \tag{2.86}$$

However, when does period 2 becomes unstable? This will occur when

$$\left|\frac{d}{dx} f(f(x))|_{x^*}\right| = 1.$$

This implies

$$\left|f'\left(x^*_-\right) f'\left(x^*_+\right)\right| = 1. \tag{2.87}$$

Let us refresh our mind the expression of $f(x)$ and its first derivative

$$\begin{aligned} f(x) &= \mu x(1-x), \\ f'(x) &= \mu - 2\mu x = \mu(1-2x). \end{aligned}$$

Now, our objective is finding the value of μ such that $f'\left(x^*_-\right) f'\left(x^*_+\right) = -1$.

$$\begin{aligned} \mu^2\left(1-2x^*_-\right)\left(1-2x^*_+\right) &= -1 \\ \mu^2\left(1-2x^*_- - 2x^*_+ + 4x^*_- x^*_+\right) &= -1. \end{aligned} \tag{2.88}$$

Referring to (2.85), the relationship between x^*_- and x^*_+ can be found,

$$x^*_- + x^*_+ = 1 + \frac{1}{\mu}, \tag{2.89}$$

$$x^*_- x^*_+ = \frac{\mu+1}{\mu^2}. \tag{2.90}$$

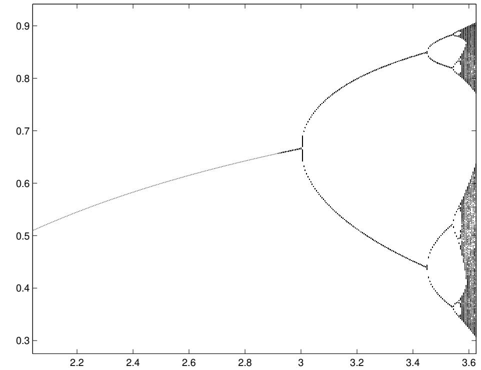

Figure 2.23: The zoom in of the bifurcation diagram of Logistic map. In this figure, we can see clearly the situation of period doubling from period 1 to period 2, then from period 2 to period 4 and so on.

Substitute them into (2.88), so

$$\begin{aligned} \mu^2\left[1 - 2\left(1 + \frac{1}{\mu}\right) + 4\left(\frac{\mu+1}{\mu^2}\right)\right] &= -1 \\ \mu^2 - 2\mu^2 - 2\mu + 4\mu + 4 &= 0 \\ -\mu^2 + 2\mu + 5 &= 0 \\ \mu &= 1 \pm \sqrt{6}. \end{aligned}$$

Reject the negative answer because μ should be a positive number, so

$$\mu = 1 + \sqrt{6} \simeq 3.44949.$$

This processes can be carried on to calculate the higher period and their stability, however, for higher period, higher complexity of the expressions are.

Feigenbaum Number

From the last session, we know that for particular values of the parameter μ, the period of the solution will be changed from 1 to 2, 2 to 4 and so on. Are there anything governing the changes of period? Do the value of μ of every changes has relationship?

We still use Logistic map to be our example. Let μ_n be the value of μ at the n^{th} bifurcation, table.2.2 shows the the first eight values of μ_n.

$\mu_1 = 3.0$	$\mu_2 = 3.449490\cdots$
$\mu_3 = 3.544090\cdots$	$\mu_4 = 3.564407\cdots$
$\mu_5 = 3.568759\cdots$	$\mu_6 = 3.569692\cdots$
$\mu_7 = 3.569891\cdots$	$\mu_8 = 3.569934\cdots$

Table 2.2: The first eight values of μ.

Upon examination of this sequence of numbers one is immediately struck by the apparent convergence. Indeed, the convergence is rapid and appears to be geometric. We denote the converged value by μ_∞. Assuming geometric convergence, we write the difference between μ_∞ and μ_n as

$$\mu_\infty - \mu_n = \frac{c}{\delta_n}, \tag{2.91}$$

where c is constant and δ is a constant > 1. A little algebra gives

$$\delta_n = \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}. \tag{2.92}$$

Using the values of μ in table.2.2, we obtain a rough estimate for

$$\delta = \lim_{n\to\infty} \delta_n = 4.6692016091\cdots,$$

which is referred to as the **Feigenbaum constant** after its discoverer (Feigenbaum, 1980).

However, what is the value of μ_∞?

$$\begin{aligned}
\delta_1 &= \frac{\mu_2 - \mu_1}{\mu_3 - \mu_2}\\
\delta_2 &= \frac{\mu_3 - \mu_2}{\mu_4 - \mu_3}\\
\delta_3 &= \frac{\mu_4 - \mu_3}{\mu_5 - \mu_4}\\
\mu_5 - \mu_4 &= \frac{\mu_4 - \mu_3}{\delta_3}\\
\mu_4 - \mu_3 &= \frac{\mu_3 - \mu_2}{\delta_2}\\
\mu_3 - \mu_2 &= \frac{\mu_2 - \mu_1}{\delta_1}
\end{aligned}$$

From these relationship, we can write

$$\mu_4 - \mu_4 = \frac{\mu_2 - \mu_1}{\delta_1\delta_2\delta_3}.$$

Since δ_1, δ_2, $\delta_3 \sim \delta$, so

$$(\mu_{n+1} - \mu_n) \propto \delta^{-(n-1)}. \tag{2.93}$$

Therefore, $\mu_{n+1} - \mu_n \to 0$ as $n \to \infty$. For logistic map, $\mu_\infty = 3.569946\cdots$. This is to remind that the Feigenbaum number is called universal because it does not depend on the details of the logistic map, but only on the fact that its iteration function has a quadratic maximum. However, μ_∞ we obtained is particular for logistic map. Here we can have a brief conclusion that for $\mu > \mu_\infty$, this implies chaos.

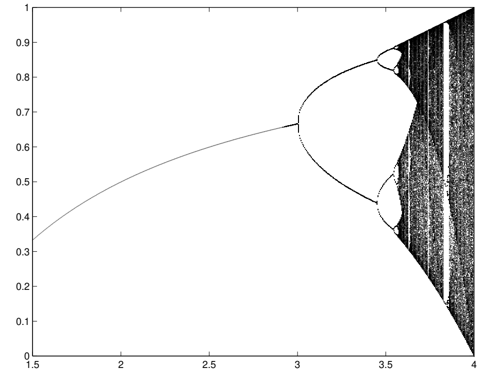

Figure 2.24: The global view of the bifurcation diagram of Logistic map.

Computation of δ (from Logisitc map)

We have just given the value of δ without works. In this session, we will discuss the computation of δ from Logistic map. The best way of computing δ is starting from the superstable fixed point, $f'(x^*) = 0$. Recall the stability at superstable fixed point: Let $x_n = x^* + \epsilon_n$, $\epsilon_n \ll 1$,

$$\begin{aligned}
x_{n+1} = f(x_{n+1}) &= f(x^* + \epsilon_n)\\
&= f(x^*) + \epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*) + \cdots,
\end{aligned}$$

Therefore,

$$x_{n+1} = x^* + \epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*), \tag{2.94}$$

$$\epsilon_{n+1} = \epsilon_n f'(x^*) + \frac{1}{2}\epsilon_n^2 f''(x^*). \tag{2.95}$$

Since x^* is a superstable fixed point, so

$$\epsilon_{n+1} = \frac{1}{2}\epsilon_n^2 f''(x^*). \tag{2.96}$$

This explain why Logistic map has quadratic convergence to the fixed point.

Suppose the map is n-cycle, such that

$$x_0 = f^{(n)}(x_0),$$

so

$$\frac{d}{dx}f^{(n)}(x_0) = f'(x_{n-1})f'(x_{n-2})\cdots f'(x_0).$$

For any n-cycle which passes through x^* with $f'(x^*) = 0$ is called superstable. This is what we have discussed before, the example is

$$f'(X^*) = 0 \quad \Rightarrow \quad \mu(1-2x^*) = 0 \quad \Rightarrow \quad x^* = \frac{1}{2}.$$

Now, we are finding the value of μ, such that $x^* = \frac{1}{2}$ is a member of the 2^{n-1}-cycle ($n = 1, 2, 3, \cdots$). We denote this value of μ by S_n and give the following definition

$$\delta = \lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\frac{S_{n-1} - S_{n-2}}{S_n - S_{n_1}}. \tag{2.97}$$

Our task is computing S_1, S_2, S_3, $\cdots$ until δ_1, δ_2, δ_3, $\cdots$ converges.

We have two approach of computing this, one is by hand and another one is numerically. First, we will talk about computing by hand. Suppose S_1 is the value of μ satisfying $\frac{1}{2} = f_\mu\left(\frac{1}{2}\right)$, therefore

$$\begin{aligned}\frac{1}{2} &= S_1\frac{1}{2}\left(1-\frac{1}{2}\right)\\ S_1 &= 2.\end{aligned}$$

Same as before, suppose S_2 is the value of μ satisfying $\frac{1}{2} = f_\mu\left(f_\mu\left(\frac{1}{2}\right)\right)$, recall the solution of $x^* = f_\mu(f_\mu(x^*))$, they are

$$x_\pm^* = \frac{1}{2\mu}\left[(\mu+1) \pm \sqrt{(\mu+1)(\mu-3)}\right].$$

Therefore

$$\begin{aligned}\frac{1}{2} &= \frac{1}{2S_2}\left[(S_2+1) + \sqrt{(S_2+1)(S_2-3)}\right]\\ 1 &= \sqrt{(S_2+1)(S_2-3)}\\ S_2^2 - 2S_2 - 4 &= 0\\ S_2 &= 1+\sqrt{5}.\end{aligned}$$

If this process is carried on, for higher index of S can be found. However, this is very time consuming. Therefore, we have use the second approach to continue this process. If we follow the definition of S_1 and S_2, so the definition of S_n should be the value of μ satisfying

$$f_\mu^{2^{n-1}}\left(\frac{1}{2}\right) = \frac{1}{2}.$$

We further define the following

$$g(\mu) = f_\mu^{2^{n-1}}\left(\frac{1}{2}\right) - \frac{1}{2}. \tag{2.98}$$

We want to find the largest root of $g(\mu)$, where all the roots are

$$\mu = S_1,\ S_2, \cdots,\ S_n.$$

However, how to solve $g(\mu) = 0$? We can use Newton's method to solve this. First we let $\mu = S_n$ be the root, and apply Taylor series to expand $g(S_n)$ about μ, so

$$0 = g(S_n) = g(\mu) + (S_n - \mu)g'(\mu) + \cdots. \tag{2.99}$$

Re-arrange (2.99), so that

$$S_n = \mu - \frac{g(\mu)}{g'(\mu)}. \tag{2.100}$$

Now, we can apply Newton's method to find S_n as

$$S_n^{(k+1)} = S_n^{(K)} - \frac{g\left(S_n^{(k)}\right)}{g'\left(S_n^{(k)}\right)}. \tag{2.101}$$

Therefore we can estimate S_n by $S_n^{(0)}$ by iterate Newton's method for $S_n^{(1)}$, $S_n^{(2)}$, $\cdots$ until converges to S_n.

However, we have 3 problems about this

1. Given a μ, how to compute $g(\mu)$ and $g'(\mu)$.
2. How do we choose $S_n^{(0)}$ close enough to S_n so that Newton's method converges to the correct root.
3. What criterion do we use for convergence?

We let $N = 2^{n-1}$ ($n = 1$ period 1, $n = 2$ period 2, $n = 3$ period 3, $\cdots$), define orbit: x_0, x_1, x_2, $\cdots$, x_{n-1} with $x_0 = \frac{1}{2}$, and

$$x_{n+1} = f_\mu(x_\mu) = \mu x_n(1-x_n).$$

Here, we want $x_N = \frac{1}{2}$. Therefore

$$g(\mu) = x_N - \frac{1}{2}. \tag{2.102}$$