Moreover, $g'(\mu)$ can be written as

$$g'(\mu) = x'_N(\mu). \tag{2.103}$$

Since $x_0 = \frac{1}{2}$ is a exact solution, therefore $x'_0 = 0$; moreover

$$x_{k+1} = \mu x_k\,(1 - x_k)\,,$$

so

$$\begin{aligned} x'_{k+1} &= x'_k\,(1 - x_k) + \mu x'_k\,(1 - x_k) - \mu x_k x'_k \\ &= x_k\,(1 - x_k) + \mu\,(1 - 2x_k)\,x'_k. \end{aligned}$$

Therefore we can write a iterative sequence such as

$$\begin{aligned} x_{k+1} &= \mu x_k\,(1 - x_k) \\ x'_{k+1} &= x_k\,(1 - x_k) + \mu\,(1 - 2x_k)\,x'_k \end{aligned} \tag{2.104}$$

with $x_0 = \frac{1}{2}$, $x'_0 = 0$, iterate $k = 1, 2, \cdots, N-1$ to find x_n and x'_n. So, we can have $g'(\mu)$.

For the second problem, if we know S_1, S_2, $\cdots$, S_n, we can compute

$$\delta_n = \frac{S_{n-1} - S_{n-2}}{S_n - S_{n-1}}.$$

We now estimate

$$\frac{S_n - S_{n-1}}{S_{n+1} - S_n} \simeq \delta_n.$$

Therefore

$$S_{n+1} - S_n \simeq \frac{S_n - S_{n-1}}{\delta_n},$$

or

$$S^{(0)}_{n+1} \simeq S_n + \frac{S_n - S_{n-1}}{\delta_n}. \tag{2.105}$$

So we use $S^{(0)}_{n+1}$ as an initial guess for μ in Newton's iteration. We can have this with the $S_1 = 2$, $S_2 = 3.236068$ and start with $\delta = 4$, while this value of δ is just a guess.

For the last problem, the process of iteration of Newton's method can be stopped, when

$$\frac{S^{(k+1)}_n - S^{(k)}_n}{S^{(k)}_n} \simeq \epsilon_{mach}, \tag{2.106}$$

ϵ_{mach} is machine epsilon, this is machine dependence, however, in general the double precision of $\epsilon_{mach} \simeq 2^{-16}$.

2.6 Fractal

To understand the Poincare section of the chaotic pendulum, we can also study the dimension of the strange attractor. By doing this, we see the property of self-similarity of the attractor.

Before this, we have to consider the classical fractal of a set.

2.6.1 Classical fractals

The most famous example of fractal is the Cantor set. Consider a line segment $[0,1]$ denoted as S_0. Then, we remove the middle one-third of this set. And, we denote the remaining part as $S_1 = [0, 1/3] \bigcup [2/3, 1]$. Another step is to remove the middle one-third of both remaining segments. This generates $S_3 = [0, 1/9] \bigcup [2/9, 3/9] \bigcup [6/9, 7/9] \bigcup [8/9, 1]$. By repeating this process to the set, we define the Cantor set as

$$S = \lim_{n \to \infty} S_n\,. \tag{2.107}$$

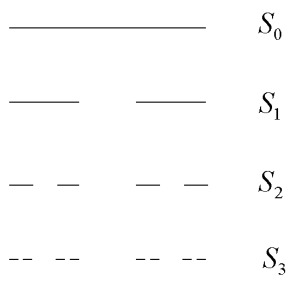

Figure 2.25: Definition of the Cantor set.

Interesting properties of the Cantor set

1. Zero length.If we denote l_n to be the length of the set S_n. We have

$$l_0 = 1\,,\; l_1 = \frac{2}{3}\,,\; l_2 = \frac{4}{9} = \left(\frac{2}{3}\right)^2\,,\; l_3 = \left(\frac{2}{3}\right)^3 \tag{2.108}$$

and etc. Therefore, we have the relation

$$l_n = \left(\frac{2}{3}\right)^n\,, \tag{2.109}$$

and this implies the length of the Cantor set, l would be

$$l = \lim_{n \to \infty} l_n = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0\,. \tag{2.110}$$

2. Countability. A set, A, is said to be countable if there exist a bijective function $f : \mathbf{N} \to A$. Therefore, using this definition, the set of integers, Z, the set of even numbers, the set of odd numbers or the set of rational numbers are all countable. If a set B is not countable, then the set is called uncountable. Therefore, the set of real numbers can be proven to be uncountable.

   Another famous example of uncountable set is the Cantor set. To prove this, we first denote any numbers between 0 and 1 by using base-3 representation, i.e. for any $x \in [0, 1]$, we write

   $$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \frac{a_4}{3^4} + \cdots , \tag{2.111}$$

   or

   $$x = 0.a_1a_2a_3a_4\cdots , \tag{2.112}$$

   with $a_n \in \{0, 1, 2\}$. Due to the property of the Cantor set, all the numbers in the Cantor set cannot have any digit "1" in the base-3 representation.

   If Cantor set is countable, then we can list out all the element by

   $$\begin{aligned} f(1) &= 0.a_{11}a_{12}a_{13}\cdots , \\ f(2) &= 0.a_{21}a_{22}a_{23}\cdots , \\ f(3) &= 0.a_{31}a_{32}a_{33}\cdots \end{aligned} \tag{2.113}$$

   and etc. However, if the Cantor set is really countable, then we let another number b where

   $$b = 0.b_1b_2b_3\cdots , \tag{2.114}$$

   where

   $$b_i \neq a_{ii} . \tag{2.115}$$

   This means if $a_{11} = 0$ then $b_1 = 2$. If $a_{11} = 2$, then b_1 equals 0. Similar case for a_{22}, a_{33} and etc. Therefore, this b will different from any element in the Cantor set. However, remember that the Cantor set includes all the numbers, x, between $[0, 1]$ such that there is no "1" for x in the base-3 representation. This number b defined earlier should therefore be included. As a result, this contradiction implies that the Cantor set is uncountable instead.

3. Self-similarity. Any small part in the Cantor set, S, will be identical to some other part in the set. For example, $A = [0, 1/2] \bigcap S$ would be similar to the set $b = [1/2, 1] \bigcap S$.

4. Dimension. The dimension of the Cantor set is not an integer. This requires the dimension of the set to be a fraction. And, we call such dimension to be fractal dimension which will be discussed in the next subsection.

2.6.2 Similarity Dimension

Suppose a self-similar set is composed of m copies of itself scaled down by a factor r. Define the similarity dimension d to be the exponent

$$m = r^d \ \text{ or } \ d = \frac{\log m}{\log r} . \tag{2.116}$$

Example 1.

Consider a line segment. If we scale the line down by a factor of 2, then there will be 2 similar copies. Therefore, the dimension d of a line is

$$d = \frac{\log 2}{\log 2} = 1 . \tag{2.117}$$

This is same as the "normal" definition of a one-dimensional object.

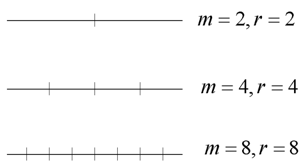

Figure 2.26: Example 1.

Example 2.

For the case of a square, if it is scaled down by a factor of 2, there will be 4 copies of similar square. This means the dimension d would be

$$d = \frac{\log 4}{\log 2} = 2 . \tag{2.118}$$

Example 3. Cantor set.

Consider the Cantor set S discussed above. If we scaled the set down by a factor of 3, only 2 similar copies will appear. Therefore,

$$d = \frac{\log 2}{\log 3} \simeq 0.6309 . \tag{2.119}$$

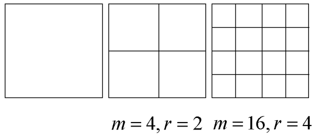

Figure 2.27: Example 2.

Example (1) and (2) show that this definition of dimension will valid for the normal definition of "dimension". This example shows, on the other hand, that the dimension of an object can be non-integer in general.

Example 4. Sierpinski's gasket.

Consider an equilateral triangle, S_0 with sides equal to l. Then, we connect the three mid-point of the three sides and the middle part of the triangle is then removed from the original triangle, S_0. The three equilateral triangles are denoted as S_1 in this step. After this, similar process is done to the three triangles respectively and this generates totally 9 equilateral triangles with sides equal $l/4$. We call this set S_2. By repeating this process, we define the Sierpinski's gasket to be the set S_∞, as shown in fig.2.28.

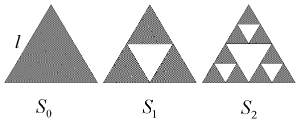

Figure 2.28: Sierpinski's Gasket.

By reducing the set by factor of 2, we have 3 similar copies of "triangles". This means

$$d = \frac{\log 3}{\log 2} \simeq 1.5850\,. \tag{2.120}$$

One more interesting thing is that, similar to the Cantor set, this uncountable set also has area equals to 0. If we let A_i to be the area of the set S_i, we

have

$$A_0 = \frac{1}{2}l\frac{\sqrt{3}}{2}l = \frac{\sqrt{3}}{4}l^2\,,\ A_1 = \frac{3}{4}A_0\,,\ A_2 = \left(\frac{3}{4}\right)^2 A_0 \text{ and } A_n = \left(\frac{3}{4}\right)^n A_0\,, \tag{2.121}$$

and this means the area, A, of the set $S = S_\infty$ would be

$$A = \lim_{n\to\infty} A_n = 0\,. \tag{2.122}$$

Example 5. Sierpinski's Carpet.

Instead of an equilateral triangle, we consider a square, S_0, with side l in this example. Similar to the above case, we remove one-ninth of the area of the square in the middle part. And, the remaining part will be denoted as S_1. By repeating this process, we have the set as shown in fig.2.29.

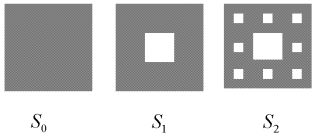

Figure 2.29: Sierpinski's Carpet.

Here, if we scale the set by factor of 3, we have 8 similar copies in this case. Therefore, from the definition of dimension, we have

$$d = \frac{\log 8}{\log 3} \simeq 1.8928\,. \tag{2.123}$$

To consider the area of the set, we first obtain the relations

$$A_0 = l^2 \text{ and } A_n = \left(\frac{8}{9}\right)^n A_0\,, \tag{2.124}$$

and this implies the area of the Sierpinski's Carpet, A, to be

$$A = \lim_{n\to\infty} A_n = 0\,. \tag{2.125}$$

Example 6. Koch Curve.

Consider a line segment, S_0, with length l. By replacing the middle one-third by 2 segments with length $l/3$, as shown in fig.2.30, we have the set S_1. Then,

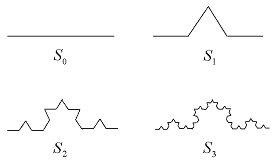

Figure 2.30: Koch Curve.

repeat this process on each small segments, we have S_2. The Koch curve is defined to be the set $S = \lim_{n\to\infty} S_n$, as shown in fig.2.30.

The dimension of this curve can be calculated by scaling the curve by factor of 3. As a result, we obtain 4 similar segments and this means the dimension of the set would be equal to

$$d = \frac{\log 4}{\log 3} \simeq 1.2619\,. \tag{2.126}$$

Example 7. Koch Snow Flake.

Instead of only a line segment, we consider a equilateral triangle as the initial state, S_0. The Koch snow flake can be obtained by repeating similar process as described in the above example, as shown in fig.2.31.

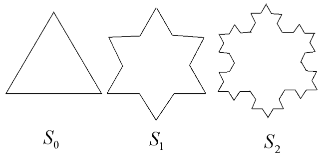

Figure 2.31: Koch Snow Flake.

A very interesting property of this example should be noticed. Consider the area bounded first. If we denote the area of S_0 to be A_0, we have

$$\begin{aligned} A_1 &= \left(1+\frac{1}{3}\right)A_0\,, \\ A_2 &= \left(1+\frac{1}{3}+\frac{1}{9}\right)A_0\,, \\ A_1 &= \left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}\right)A_0\,. \end{aligned} \tag{2.127}$$

This means the area bounded would be converging to a finite number. However, on the other hand, the length of the boundary would be diverging. Infinite length of a line bounded finte area.

2.6.3 Correlation Dimension

However, in general, fractal will not be completely similar everywhere. Therefore, in order to compute the fractal dimension of a strange attractor, we define the dimension of a fractal object by using correlation dimension, as described in reference [?].

We first generate points, $\{\vec{x}_n, i = 1, 2, \cdots, N\}$, lying in a fractal set, A. Then, we fix a point $\vec{x}$ in this set. By letting $N_{\vec{x}}(\epsilon)$ be the number of points on A inside a ball of radius ϵ centered at $\vec{x}$, i.e. $N_{\vec{x}}(\epsilon)$ represents the number of points in the attractor A such that the distance between those points and $\vec{x}$ is less than ϵ.

Then, we have the relation

$$N_{\vec{x}}(\epsilon) \propto \epsilon^{d_{\vec{x}}}\,, \tag{2.128}$$

with $d_{\vec{x}}$ denotes the pointwise dimension at the point $\vec{x}$.

Further, we define $C(\epsilon)$ be the average values of all the $N_{\vec{x}}(\epsilon)$ at the all the points $\vec{x}$ in A, i.e.

$$C(\epsilon) = \langle N_{\vec{x}}(\epsilon)\rangle\,. \tag{2.129}$$

Then, the correlation dimension of a fractal object d would satisfy

$$C(\epsilon) \propto \epsilon^d\,. \tag{2.130}$$

Then, the correlation dimension d can be obtained as

$$d = \frac{\log[C(\epsilon)]}{\log(\epsilon)}\,. \tag{2.131}$$

Example 1.

If the point x consists of all real numbers between $0 < x < 1$, then d should still be 1. Fig.2.32a shows the plot of "loglog($\epsilon, C(\epsilon)$)" (in the language of MATLAB) of a line from 0 to 1. From the graph, by computing the slope of the line, the dimension, d, can also be obtained. In this numerical experiment, we found that the dimension of a line is 0.9978, which is very similar to 1 as we can predict.

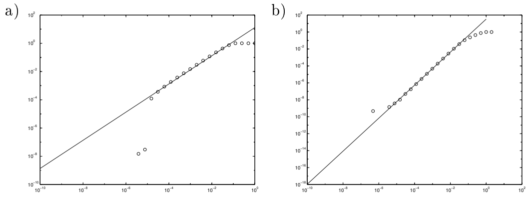

Figure 2.32: Calculation of the fractal dimension of a line and a plane.

Example 2.

If the points $\vec{x} = (x, y)$ consist of all real numbers between $0 < x < 1$ and $0 < y < 1$, then d should also be equal to 2. In our calculation, we found that the dimension is approximately equal to 1.9488, as shown in fig.2.32b. This case and the above one can be generated by producing x_i (and y_i) for $i = 1, \cdots, N$ using a random number generator.

Example 3. Logistic Map.

As describe in the early section, Logistic Map, given by $x_n = \mu x_{n-1}(1 - x_{n-1})$, would become chaotic when $\mu = \mu_\infty \simeq 3.56994567120529687$. Here, we calculate the fractal dimension of the strange attractor at this μ_∞ and a value tiny different from this value, say $\mu = 3.58$. Fig.2.33 shows the procedure of calculating the fractal dimension of the Logistic Map at different values of μ. When we are choosing 2^{15} points in the attractor, we found that the fractal dimension of the strange attractor when $\mu = 3.56994567$ and $\mu = 0.8953$ are approximately 0.5143 and 0.8953 respectively.

2.6.4 Linear Regression

When we are calculating the value of d from the computational experimental data, we have to approximate the slope of a "underlying" line. In order to minimize the error of this approximation, we have to fit a line to the set of data such that the sum of squares of the error is minimized. This can be done by introducing the idea of Least Square Fitting.

Consider a set of data $\{(x_i, y_i), i = 1, 2, \cdots, n\}$ and we want to fit a line with the form of

$$y = ax + b \tag{2.132}$$

to this set of numbers such that the errors between the line and the data are

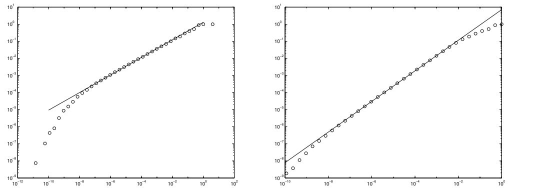

Figure 2.33: Calculation of the fractal dimension of the Logistic Map at (a) $\mu = 3.56994567$ and (b) $\mu = 3.58$.

least. Then, the problem is to minimize the value of

$$\begin{aligned} S &= \sum_{i=1}^{n} e_i^2 \\ &= \sum_{i=1}^{n} (y_i - a x_i - b)^2 \,, \end{aligned} \tag{2.133}$$

with e_i denotes the difference between the experimental value y_i with the value of the fitted line at the point x_i. In order to minimize this value, it requires that

$$\frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = 0 \,. \tag{2.134}$$

This implies that

$$\begin{aligned} a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} x_i y_i \\ a\sum_{i=1}^{n} x_i + bn &= \sum_{i=1}^{n} y_i \,, \end{aligned} \tag{2.135}$$

and a and b can be found to be

$$\begin{aligned} a &= \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n(\sum x_i^2)\left(\sum x_i\right)^2} \\ b &= \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{n(\sum x_i^2)\left(\sum x_i\right)^2} \,. \end{aligned} \tag{2.136}$$

Appendix - Variation of Phase Volume

In fact, it should not must be volume, for 2D case, it should be area. The method of determination of whether the phase volume has variation can be

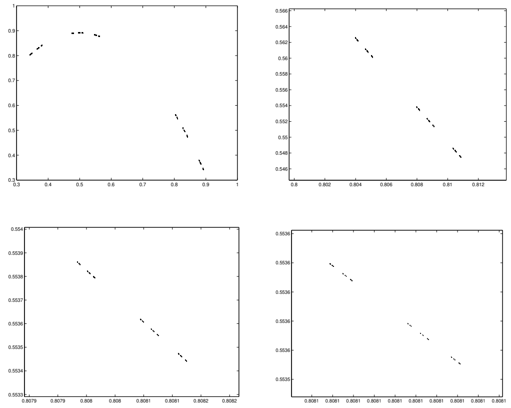

Figure 2.34: The plot of x_i versus x_{i+1} when $\mu = 3.56994567$.

applied in higher dimension. However, for convenience, we will use 3D case to be our example.

Here we have a dynamical system:

$$\begin{aligned}
\dot{x} &= F_1(x,y,z) \\
\dot{y} &= F_2(x,y,z) \\
\dot{z} &= F_3(x,y,z)
\end{aligned}$$

Consider a small volume of phase space, $\delta\Gamma$, such that

$$\delta\Gamma = \delta x\, \delta y\, \delta z. \tag{2.137}$$

Here,

$$\begin{aligned}
\delta x &= x_1 - x_0 \\
\delta y &= y_1 - y_0 \\
\delta z &= z_1 - z_0
\end{aligned}$$

How does δx changes in time? Let's see what happen in x,

$$\begin{aligned}
x_1(t+\Delta t) &= x_1(t) + \Delta t F_1(x_1,y,z), \\
x_0(t+\Delta t) &= x_0(t) + \Delta t F_1(x_0,y,z).
\end{aligned}$$

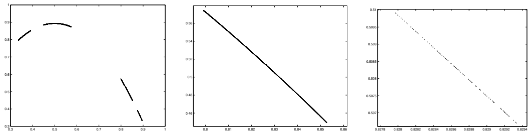

Figure 2.35: The plot of x_i versus x_{i+1} when $\mu = 3.58$.

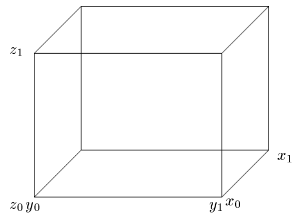

Figure 2.36: The setting of a box.

Therefore

$$\delta x(t+\Delta t) = \delta x(t) + \Delta t \left[F_1(x_1,y,z) - F_1(x_0,y,z)\right]. \tag{2.138}$$

So,

$$\begin{aligned}
\frac{\delta x(t+\Delta t) - \delta x(t)}{\Delta t} &= \frac{d}{dt}\delta x \\
&= F_1(x_0 + \delta x, y, z) - F_1(x_0,y,z) \\
&= \delta x \frac{\partial F_1}{\partial x} \\
\frac{d}{dt}\delta x &= \delta x \frac{\partial F_1}{\partial x}.
\end{aligned}$$

Perform this on y and z, so

$$\frac{d}{dt}\delta y = \delta y \frac{\partial F_2}{\partial y}, \qquad \frac{d}{dt}\delta z = \delta z \frac{\partial F_3}{\partial z}.$$

Then, we can write the following

$$\frac{d}{dt}(\delta\Gamma) = \left(\frac{d}{dt}\delta x\right)\delta y\,\delta z + \delta x\left(\frac{d}{dt}\delta y\right)\delta z + \delta x\,\delta y\left(\frac{d}{dt}\delta z\right)$$

$$
\begin{aligned}
&= \delta x\, \delta y\, \delta z \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) \\
&= \delta\Gamma \left(F_{1x} + F_{2y} + F_{3z} \right).
\end{aligned}
$$

Therefore,

$$\frac{d}{dt}(\delta\Gamma) = (\delta\Gamma)\left(\vec{\nabla} \cdot \vec{F}\right). \tag{2.139}$$

Now, we can have a conclusion about the variation of phase volume, if

$$
\begin{aligned}
&\vec{\nabla} \cdot \vec{F} = 0, && \text{phase space volume is conserved,} \\
&\vec{\nabla} \cdot \vec{F} < 0, && \text{phase space volume contracts.}
\end{aligned}
$$

Let's use the unforced pendulum and forced pendulum as examples to illustrate this concept. For unforced pendulum,

$$
\begin{aligned}
\dot{\theta} &= u = F_1(\theta, u) \\
\dot{u} &= -\frac{1}{q}u - \theta = F_2(\theta, u).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{F} &= F_{1\theta} + F_{2u} \\
&= 0 + \left(-\frac{1}{q}\right) \\
&= -\frac{1}{q}.
\end{aligned}
$$

So,

$$
\begin{aligned}
q \to \infty,\ \text{undamped case} &\quad\Rightarrow\quad \text{phase space volume } \textbf{conserves}, \\
0 < q < \infty,\ \text{damped case} &\quad\Rightarrow\quad \text{phase space volume } \textbf{contracts}.
\end{aligned}
$$

How about forced pendulum?

$$
\begin{aligned}
\dot{\theta} &= u = F_1(\theta, u, \phi) \\
\dot{u} &= -\frac{1}{q}u - \theta + g\cos\phi = F_2(\theta, u, \phi) \\
\dot{\phi} &= \omega_D = F_3(\theta, u, \phi)
\end{aligned}
$$

We examine the gradient of $\vec{F}$, so

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{F} &= F_{1\theta} + F_{2u} + F_{3\phi} \\
&= 0 + \left(-\frac{1}{q}\right) + 0 \\
&= -\frac{1}{q}.
\end{aligned}
$$

The result is same as the unforced case, so the same conclusion can be applied in this case.

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