Chapter 3

Topics In Computational Fluid Dynamics

3.1 Notations Used In This Chapter

$\vec{x}$	Position vector
$\vec{u}(\vec{x}, t)$	Fluid velocity at the point $\vec{x}$ at time t
$P(\vec{x}, t)$	Normal force per unit area at the point $\vec{x}$ at time t
$\rho(\vec{x}, t)$	Fluid density (mass / unit volume)
ν	Kinematic viscosity, assumed to be constant for a fluid
ψ	Stream function
ω	Vorticity
Re	Reynolds number

Several points should be notice here:

The velocity vector of the fluid $\vec{u}$ would be written as (u, v, w) or sometimes $u\hat{x} + v\hat{y} + w\hat{z}$ for convenient.

$\rho(\vec{x}, t)$ will preceicely be a function of space and time for real fluids. However, for simplification, ρ will be treated as a constant and the fluid will be called **incompressible** fluid. This is a reasonable approximation for most of the fluid. Therefore, for the rest of the chapter, we will write ρ as an constant which would still vary among different fluids.

Beside ρ, the kinematic viscosity ν would also be treated as a **characteristic varible** of the fluid. This means the quantities will depend on the fluid itself, instead of the variations in the position and time of a particular situation. On the other hand, $\vec{u}(\vec{x}, t)$ and $P(\vec{x}, t)$ are called the **dynamic variables** for the fluid at different situations. This means these quantities will be changed fro different locations, time and the fluid itself.

3.2 Derivations Of The Navier-Stokes Equations

3.2.1 Continuity Equation

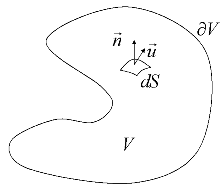

Figure 3.1: Derivation of Continuity Equation.

Continuity equation states the conservation of mass. Consider any small amount of fluid in a control volume V which is fixed in space. Then, the total mass in V would be

$$\int\int\int_V \rho dV\,. \tag{3.1}$$

Then, the rate of change of the total mass in that control volume would be equal to

$$\frac{d}{dt}\int\int\int_V \rho dV\,. \tag{3.2}$$

And, this amount would be balanced by the flow across it's boundary ∂V.

The volume flux passes through a small surface element dS during some time dt would be equal to

$$\vec{u}dt \cdot \vec{n}dS\,, \tag{3.3}$$

where $\vec{n}$ represents the normal vector of this surface element. Using this, the mass flux through the same surface during dt would be

$$\rho\vec{u} \cdot \vec{n}dSdt\,. \tag{3.4}$$

As a result, the mass **influx** on dS **per unit time** would become

$$-\rho\vec{u} \cdot \vec{n}dS\,, \tag{3.5}$$

and the mass influx through ∂V per unit time would be written as

$$-\int\int_{\partial V} \rho\vec{u} \cdot \vec{n}dS\,. \tag{3.6}$$

Then, mathematically the conservation of mass can be written as

$$\frac{d}{dt}\int\int\int_V \rho dV = -\int\int_{\partial V} \rho\vec{u}\cdot\vec{n}dS\,. \tag{3.7}$$

If ρ and $\vec{u}$ are both continuous, then by using the Divergence Theorem in multi-variable calculus, the above equation can be simplified to be

$$\int\int\int_V \frac{\partial\rho}{\partial t}dV + \int\int\int_V \nabla\cdot(\rho\vec{u})dV = 0\,. \tag{3.8}$$

Because V is an arbitrary, the continuity equation can be written as

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0\,,} \tag{3.9}$$

everywhere in V.

3.2.2 Momentum Equation

The momentum equation is another form of Newton's second law of motion. In this case, we are studying the forces acting on the fluid. By Newton's second law, the total force on the control volume can be written as the density times the acceleration. However, the definition of the acceleration should be a little bit different. Normally

$$\vec{a} = \frac{\partial\vec{u}}{\partial t}\,. \tag{3.10}$$

But, in the description of the fluid motion, this kind of definition would be corresponding to comparing the change in the velocity of **two different** control volume passing through the same location. This is clearly not the definition we want. Therefore, the definition of acceleration must be redefined here.

Because we are considering the rate for change of the velocity of the same control volume,

$$\begin{aligned}
\vec{a} &= \frac{\vec{u}(x+\Delta x, y+\Delta y, z+\Delta z) - \vec{u}(x,y,z)}{\Delta t} \\
&= \frac{\Delta x}{\Delta t}\frac{\partial\vec{u}}{\partial x} + \frac{\Delta y}{\Delta t}\frac{\partial\vec{u}}{\partial y} + \frac{\Delta z}{\Delta t}\frac{\partial\vec{u}}{\partial z} + \frac{\partial\vec{u}}{\partial t} \\
&= u\frac{\partial\vec{u}}{\partial x} + v\frac{\partial\vec{u}}{\partial y} + w\frac{\partial\vec{u}}{\partial z} + \frac{\partial\vec{u}}{\partial t} \\
&= (\vec{u}\cdot\nabla)\vec{u}\,.
\end{aligned} \tag{3.11}$$

The force on the fluid, on the other hand, can be classified into two types, pressure forces and viscous forces.

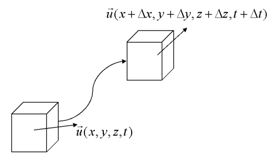

Figure 3.2: Definition of acceleration.

Pressure Forces, $\vec{f}_P$

This force is due to the pressure by the adjacent control volume cell. The y-component of this force **per unit volume** would be equal to

$$\frac{PA-(P+dP)A}{V} = \frac{PA-(P+dP)A}{Ady} = -\frac{\partial P}{\partial y}\,. \tag{3.12}$$

With the other components of the forces, the pressure forces can be written as

$$\vec{f}_P = -(P_x, P_y, P_z) = -\nabla P\,. \tag{3.13}$$

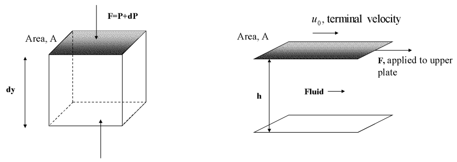

Figure 3.3: Pressure Force and Viscous Force.

Viscous Forces, $\vec{f}_\nu$

If we let μ to be the coefficient of viscosity of the fluid, u_0 to be the terminal velocity of the upper plate, A to be the area of the plate and F to be the applied force to the upper plate, we have

$$\frac{F}{A} = \mu\frac{u_0}{d} \simeq \mu\frac{du}{dy}\,, \tag{3.14}$$

for small u_0 and d.

The general form of the **shear stresses** for incompressible fluid can be written as

$$\tau_{ij} = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\,. \tag{3.15}$$

Therefore, the i component of the viscous forces can be written as

$$f_{\nu;i} = \frac{\partial \tau_{ij}}{\partial x_j} = \nu\left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i \partial x_j}\right)\,. \tag{3.16}$$

Because

$$\frac{\partial^2 u_j}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0\,, \tag{3.17}$$

the viscous forces can be written as

$$\vec{f}_\nu = \mu\nabla^2\vec{u}\,. \tag{3.18}$$

Momentum Equation

By combining these results, the momentum equation can be expressed as

$$\rho\left[\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right] = -\nabla P + \mu\nabla^2\vec{u}\,. \tag{3.19}$$

If we further define the kinematic viscosity, ν to be μ/ρ, we have

$$\boxed{\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = \frac{-1}{\rho}\nabla P + \nu\nabla^2\vec{u}\,.} \tag{3.20}$$

3.2.3 Non-dimensionalization

Notice that all the variables in the expression are having units, to write the equation in a nicer form, we choose the units of the variables to be

$$\vec{u} = u_0\vec{u}'\,,\ \vec{x} = l_0\vec{x}'\,,\ t = \frac{l_0}{u_0}t' \ \text{ and }\ \frac{P}{\rho} = u_0^2 P'\,. \tag{3.21}$$

By doing this we can obtain the dimensionless for of the Continuity equation,

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0\,,} \tag{3.22}$$

and the Momentum equation,

$$\boxed{\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\nabla P + \frac{1}{Re}\nabla^2\vec{u}\,,} \tag{3.23}$$

with

$$Re = \frac{u_0 l_0}{\nu}\,. \tag{3.24}$$

This Re is called the Reynolds number of the system and this number shows important property in Section (2.5) when we solve the problem of flow passing through a rectangular obstacle numerically.

3.3 Laminar Solution Of The Navier-Stokes Equations

From the above section, we obtained the following Navier-Stokes equation,

$$\frac{\partial u_i}{\partial t} + u_i\frac{\partial u_i}{\partial x_j} = \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j \partial y_j}\,. \tag{3.25}$$

Although the equations are highly non-linear, for some special cases, we can still solve it analytically. Following shows several cases in which the non-linear terms can be removed.

3.3.1 Plate Couette Flow

We are considering the fluid flow between two plates with separating distance d. The lower one is fixed on the bottom, while the upper one is moving to the right with constant velocity u_0. Then, we are looking for a steady solution with constant P in between the plates and in the form of

$$\vec{u}(x,y,z) = (u(y),0,0)\,. \tag{3.26}$$

Therefore, we have

$$\frac{\partial u_i}{\partial t} = 0\,,\ \frac{\partial P}{\partial x_i} = 0 \ \text{ and }\ u_j\frac{\partial u_i}{\partial x_j} = 0\,. \tag{3.27}$$

As a result, the Navior-Stokes equations can be simplified to be

$$\nu \frac{\partial^2 u(y)}{\partial y^2} = 0 \,. \tag{3.28}$$

And, the solution can be written as

$$u(y) = \frac{u_0}{d} y \,, \tag{3.29}$$

by using suitable boundary conditions described above.

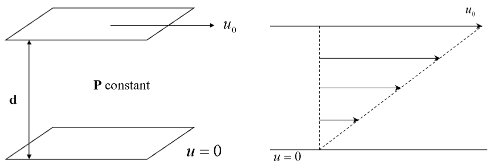

Figure 3.4: Couette Flow and it's solution.

3.3.2 Channel Flow Or Poiseuille Flow

In this situation, we are considering the fluid flows between two fixed plates, i.e. $u = 0$. The distance between these two plates would be equal to d. The pressure at one end would be P_t, which is larger than that at the other end, P_b. Then, we look for solution in the same form as in the previous case,

$$\vec{u} = (u(y), 0, 0) \,. \tag{3.30}$$

Then, the Navier-Stokes equations can be simplified to be

$$\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} = 0 \,, \; \frac{1}{\rho}\frac{\partial P}{\partial y} = 0 \text{ and } \frac{1}{\rho}\frac{\partial P}{\partial z} = 0 \,. \tag{3.31}$$

From the second and the third equations, we can see that P would be independent of y and z. Therefore, for constant rate of change in the pressure along x-direction,

$$G = \frac{\partial P}{\partial x} = \text{ constant} \,. \tag{3.32}$$

And, we can solve u from the equation deduced from the first simplified Navier-Stokes equations. By using the boundary conditions

$$u(0) = 0 \text{ and } u(d) = 0 \,, \tag{3.33}$$

we can have

$$u(y) = \frac{G}{2\rho\nu} y(d - y) \,. \tag{3.34}$$

From this, the maximum velocity of the fluid flow would then be obtained as

$$u_{\max} = \frac{Gd^2}{8\rho\nu} \,, \tag{3.35}$$

when $y = d/2$.

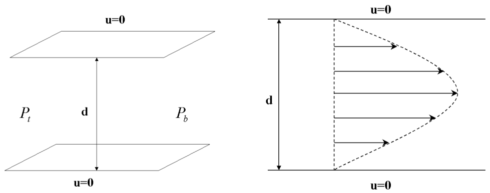

Figure 3.5: Channel Flow.

3.3.3 Pipe Flow

Instead of considering fluid in two plates, we are considering the fluid in a pipe with circular cross section. Same as in the above case, pressure on the end would be greater than the other end. Then, we are looking for solution with P depends on x only and

$$\vec{u} = (u(y, z), 0, 0) \,. \tag{3.36}$$

Then, by using two of three Navier-Stokes equations, we can have

$$G = \frac{\partial P}{\partial x} = \text{ constant } \,, \tag{3.37}$$

and, by substitude this relation to the remaining equation, we obtain

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-G}{\nu\rho} \,. \tag{3.38}$$

In this situation, it would be easier to solve the equation by using polar coordinates instead if we consider a solution independent of θ. Then, the Laplacian can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) \,, \tag{3.39}$$

and the equation can be transformed into

$$\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})u = \frac{-G}{\nu\rho}r\,. \tag{3.40}$$

By solving this equation subjected to the boundary conditions

$$u(R) = 0 \text{ and } u(0) < \infty\,, \tag{3.41}$$

with R equals to the radius of the pipe, the solution can be obtained to be

$$u = \frac{G}{4\nu\rho}R^2\left[1 - \left(\frac{r}{R}\right)^2\right]\,, \tag{3.42}$$

with

$$u\text{max} = \frac{G}{4\nu\rho}R^2\,, \tag{3.43}$$

at $r = 0$.

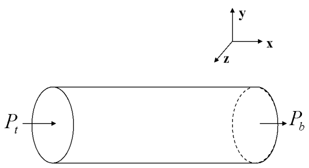

Figure 3.6: Pipe Flow.

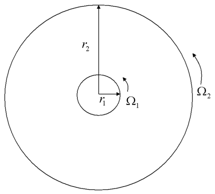

Figure 3.7: Taylor Couette Flow.

3.3.4 Taylor Couette Flow

Consider an incompressible flow between two concentric circular cylinders of radii r_1 and r_2, rotating about their common axes with angular velocities Ω_1 and Ω_2, as shown in fig.3.7.

By using cylindrical coordinates, i.e.

$$v = v(x,\sigma,\phi) = (v_x, v_\sigma, v_\phi)\,, \tag{3.44}$$

and assuming

$$v_x = v_\sigma = 0\,,\; v_\phi = v_\phi(\sigma) \text{ and } P = P(\sigma)\,, \tag{3.45}$$

the Navier-Stokes equations can be written as

$$\begin{aligned}
\frac{\partial v_x}{\partial x} + \frac{1}{\sigma}\frac{\partial(\sigma v_\sigma)}{\partial \sigma} + \frac{1}{\sigma}\frac{\partial v_\phi}{\partial \phi} &= 0 \\
\frac{\partial v_x}{\partial t} + \vec{v}\cdot\nabla v_x &= \frac{-1}{\rho}\frac{\partial P}{\partial x} + \gamma\nabla^2 v_x \\
\frac{\partial v_\sigma}{\partial t} + \vec{v}\cdot\nabla v_\sigma - \frac{v_\phi^2}{\sigma} &= \frac{-1}{\rho}\frac{\partial P}{\partial \sigma} + \gamma\left(\nabla^2 v_\sigma - \frac{v_\sigma}{\sigma^2} - \frac{2}{\sigma^2}\frac{\partial v_\phi}{\partial \phi}\right) \\
\frac{\partial v_\phi}{\partial t} + \vec{v}\cdot\nabla v_\phi + \frac{v_\rho v_\phi}{\rho} &= \frac{-1}{\rho\sigma}\frac{\partial P}{\partial \phi} + \gamma\left(\nabla^2 v_\phi + \frac{2}{\sigma^2}\frac{\partial v_\sigma}{\partial \phi} - \frac{v_\phi}{\partial \sigma^2}\right)
\end{aligned}$$

or

$$\frac{\partial v_\phi}{\partial \phi} = 0\,, \tag{3.46}$$

$$\frac{-v_\phi^2}{\sigma} = \frac{-1}{\rho}\frac{\partial P}{\partial \sigma}\,, \tag{3.47}$$

$$\frac{\partial v_\phi}{\partial t} + \vec{v}\cdot\nabla v_\phi = \gamma\left(\nabla^2 v_\phi - \frac{v_\phi}{\sigma^2}\right)\,. \tag{3.48}$$

For steady flow, (3.48) becomes

$$0 = \gamma\left[\frac{1}{\sigma}\frac{\partial}{\partial \sigma}\left(\sigma\frac{\partial v_\phi}{\partial \sigma}\right) - \frac{v_\phi}{\sigma}\right]\,, \tag{3.49}$$

and by solving this equation, we have

$$v_\phi = A\sigma + \frac{B}{\sigma}\,. \tag{3.50}$$

By using the boundary conditions

$$v_\phi(r_1) = r_1\Omega_1 \text{ and } v_\phi(r_2) = r_2\Omega_2\,, \tag{3.51}$$

the constants A and B can be calculated to be

$$A = \frac{r_1^2\Omega_1 - r_2^2\Omega_2}{r_1^2 - r_2^2} \text{ and } B = \frac{r_1^2 r_2^2(\Omega_2 - \Omega_1)}{r_1^2 - r_2^2}\,. \tag{3.52}$$

And therefore,

$$v_\phi(\sigma) = \frac{r_1^2\Omega_1 - r_2^2\Omega_2}{r_1^2 - r_2^2}\sigma + \frac{r_1^2 r_2^2(\Omega_2 - \Omega_1)}{r_1^2 - r_2^2}\frac{1}{\sigma}\,. \tag{3.53}$$

3.4 Stream Function - Vorticity Equation For 2D Flow

3.4.1 Streamlines

A streamline at a particular time t can be defined to be a curve whose tangent is everwhere parallel to the velocity vector, i.e. $\vec{u} || d\vec{x}$ or

$$\vec{u} \times d\vec{x} = 0 \text{ or } \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u & v & w \\ dx & dy & dz \end{vmatrix} = 0 \,. \tag{3.54}$$

This means

$$\frac{dx}{u(x,y,z,t)} = \frac{dy}{v(x,y,z,t)} = \frac{dz}{w(x,y,z,t)} \,, \tag{3.55}$$

which consists of **two useful ordinary differential equations** with t as a parameter.

A stream surface would then be defined to be a surface composed of streamlines. This means if the surface equation is $S(x,y,z,t) = 0$, then this surface is a stream surface if and only if

$$\vec{u} \cdot \nabla S = 0 \text{ on } S = 0. \tag{3.56}$$

If this surface forms a tube shape, this is also called a streamtube.

There are several characteristics for the streamlines.

1. Two streamlines cannot intersect, except when the case of $\vec{u} = 0$.

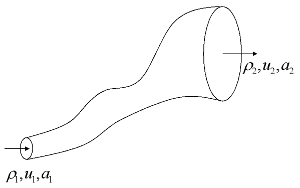

Figure 3.8: Streamlines far apart imply slow speed.

2. When the streamlines close together, the fluid in the region is moving in high speed. When the streamlines are far apart, the velocity of the fluid is low. This can be shown in this example:

Example 1.

Consider a streamtube with vfluid moving in the tube on one end having density ρ_1, speed u_1 and cross section area a_1. and, the fluid is moving out from the tube with density ρ_2, speed u_2 and cross section area a_2 on the other end. Because of the property (1),

$$\begin{aligned} \text{Mass going in per unit time} &= \text{Mass going out per unit time} \\ \rho_1 u_1 a_1 &= \rho_2 u_2 a_2 \,. \end{aligned} \tag{3.57}$$

For incompressible flow, this means $u \propto a^{-1}$. Therefore, for the same number of streamlines crossing the surface with area a_1 and a_2, higher density of streamlines correspond to faster speed of the fluid.

In two dimensional incompressible flow, the continuity equation in cartesian coordinates can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \,. \tag{3.58}$$

This implies that there exists a function $\psi = \psi(x,y,t)$ such that

$$u = \frac{\partial \psi}{\partial y} \,, \; v = -\frac{\partial \psi}{\partial x} \,. \tag{3.59}$$

Because the line $\psi(x,y,t) = c$ for any constant c is a streamline at time t, this function ψ is called the stream function. And, therefore, the contour plot of this stream function ψ would become a plot of streamlines. This is an important relationship we will use.

Example 2.

Consider the flow with velocities

$$u = \frac{-ky}{x^2+y^2} \text{ and } v = \frac{kx}{x^2+y^2} \,, \tag{3.60}$$

for some positive number k. Then, by solving the equation

$$\frac{dx}{-ky/(x^2+y^2)} = \frac{dy}{kx/(x^2+y^2)} \,, \tag{3.61}$$

we have the streamlines

$$x^2 + y^2 = c \,, \tag{3.62}$$

for any constant c, as shown in fig.3.9.

Example 3.

In two dimensional flow, we define $z = x + iy$, with $i = \sqrt{-1}$, and the **complex velocity potential** $W(z)$ such that the imaginary part of W represents the stream function of the flow, i.e.

$$\psi = \Im(W) \,. \tag{3.63}$$

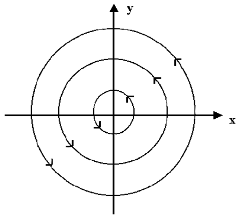

Figure 3.9: Stream lines of Example 2.

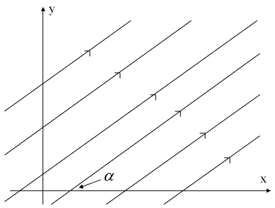

Figure 3.10: Streamlines for Example 3.

Consider

$$W(z) = Ue^{-i\alpha}z\,, \tag{3.64}$$

the stream function is therefore

$$\psi = Ue^{-i\alpha}y\,. \tag{3.65}$$

The solution of streamlines are plotted on fig.3.10.

Example 4.

If the complex velocity potential becomes

$$W = \frac{Q}{2\pi}\ln z \;\; Q \in \Re, \tag{3.66}$$

We have

$$\psi = \frac{Q}{2\pi}\theta\,, \tag{3.67}$$

as shown in fig.3.11.

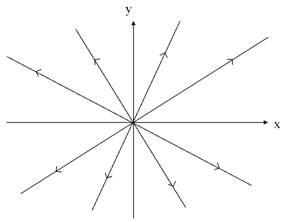

Figure 3.11: Streamlines for Example 4.

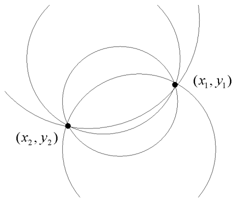

Figure 3.12: Streamlines for Example 5.

Example 5.

Consider

$$W = \frac{Q}{2\pi}\ln(z - z_1) - \frac{Q}{2\pi}\ln(z - z_2)\,, \tag{3.68}$$

then, the imaginary part of W can be determined to be

$$\frac{Q}{2\pi}\arg\left(\frac{z - z_1}{z - z_2}\right)\,. \tag{3.69}$$

Therefore, we have the streamlines are the family of circles passing through $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

3.4.2 Vorticity

The vorticity vector $\vec{\omega}$ is defined to be $\nabla \times \vec{u}$. It is a measure of the strength of the rotation of the fluid or the strength of the swirl. For two dimensional flow, the vorticity vector can also be written as

$$\begin{aligned} \vec{\omega} &= \nabla \times \vec{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix} \\ &= (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\hat{z} = \omega \hat{z}\,. \end{aligned} \tag{3.70}$$

If $\vec{\omega} = \vec{0}$, the fluid is called irrotational. Here are several examples:

Example 1

Uniform flow of the fluid can be expressed as $\vec{u}$ =constant. This implies immediately that $\vec{\omega} = 0$. This means the fluid flow is irrotational.

Example 2

Consider a free vortex flow in which

$$u = \frac{-ky}{x^2 + y^2}\,,\; v = \frac{kx}{x^2 + y^2} \;\text{ and }\; w = 0\,, \tag{3.71}$$

for some positive real number k representing the strength of the vertex. Then, the streamlines can be shown to be circles centered at the origin. By calculating the vorticity vector, we can find that $\vec{\omega}$ also equals to zero even the streamlines are all circle.

Example 3

Consider the following Shear flow with velocities

$$u = ky \;\text{ and }\; v = w = 0\,. \tag{3.72}$$

Therefore,

$$\vec{\omega} = (0 - k)\hat{z} \neq \vec{0} \;\text{ for } k \neq 0. \tag{3.73}$$

Here in this case, the flow is rotational even when the streamlines are all straight lines.

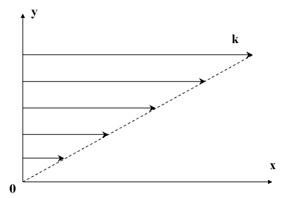

Figure 3.13: Flow vectors of Shear Flow in Example 3.

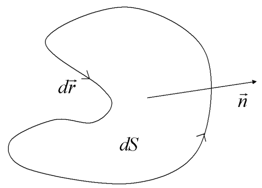

Figure 3.14: Circulation.

3.4.3 Circulation

As we can see from the above examples, the idea of vorticity not quite depends on the shape of the streamlines. Here, we introduce the idea of circulation which is closely related to the streamline on the other hand. The circulation Γ around a closed contour c is defined by

$$\Gamma = \oint_C \vec{u} \cdot d\vec{r}\,. \tag{3.74}$$

Applying Stokes Theorem to this definition, we have

$$\Gamma = \int\int_S (\nabla \times \vec{u}) \cdot \vec{n} dS = \int\int_S \vec{\omega} \cdot \vec{n} dS\,. \tag{3.75}$$

This is a relationship between the circulation and the vorticity of the flow. From this, a flow is irrotational if and only of $\Gamma = 0$ for every contour C.

Example 1.

Consider a contour $C = C_1 + C_2 + C_3 + C_4$ in the Shear flow. Therefore,

$$\begin{aligned} \Gamma &= \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \\ &= 0 + 0 + \Gamma_3 + 0 \neq 0\,, \end{aligned} \tag{3.76}$$

and as a results, $\omega \neq 0$.

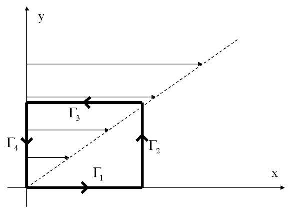

Figure 3.15: Flow vectors of Shear Flow in Example 1.

3.4.4 Vorticity Equation

By taking curl, $\nabla\times$, to the momentum equations, we have

$$\nabla \times \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right] = \nabla \times \left(\frac{-1}{\rho}\nabla P + \nu\nabla^2\vec{u}\right)\,. \tag{3.77}$$

Because "curl of gradient" equals to zero and

$$\begin{aligned} \nabla \times \left(\frac{\partial \vec{u}}{\partial t}\right) &= \frac{\partial}{\partial t}(\nabla \times \vec{u}) = \frac{\partial \vec{\omega}}{\partial t} \\ \nabla \times \left(\frac{-1}{\rho}\nabla P\right) &= 0 \\ \nabla \times (\nu\nabla^2\vec{u}) &= \nu\nabla^2(\nabla\times\vec{u}) = \nu\nabla^2\vec{\omega}\,, \end{aligned} \tag{3.78}$$

the expression can be simplified to

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{u}\cdot\nabla)\vec{u} = \nu\nabla^2\vec{\omega}\,. \tag{3.79}$$

Consider the identity

$$\vec{u} \times (\nabla \times \vec{u}) = \frac{1}{2}\nabla(\vec{u}\cdot\vec{u}) - (\vec{u}\cdot\nabla)\vec{u}\,, \tag{3.80}$$

we have

$$(\vec{u}\cdot\nabla)\vec{u} = \frac{1}{2}\nabla(\vec{u}\cdot\vec{u}) - \vec{u}\times\vec{\omega}\,. \tag{3.81}$$

By taking curl of this relation, we have

$$\nabla \times [(\vec{u}\cdot\nabla)\vec{u}] = -\nabla \times (\vec{u}\times\vec{\omega}) = \nabla \times (\vec{\omega}\times\vec{u})\,. \tag{3.82}$$

Then, the vorticity equation can be written as

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{\omega}\times\vec{u}) = \nu\nabla^2\vec{\omega}\,, \tag{3.83}$$

or

$$\underbrace{\frac{\partial \vec{\omega}}{\partial t} + \vec{u}\cdot\nabla\vec{\omega}}_{\text{substantive derivative}} = \underbrace{\vec{\omega}\cdot\nabla\vec{u}}_{\text{vortex streching}} + \underbrace{\nu\nabla^2\vec{\omega}}_{\text{viscous diffusion}}\,. \tag{3.84}$$

3.4.5 Stream Function And Vorticity Formulation For 2-D Steady Flow

From the definitions of the streamlines and vorticity, we have obtained the relations

$$u = \frac{\partial \psi}{\partial y}\,,\; v = -\frac{\partial \psi}{\partial x} \;\text{ and }\; \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,, \tag{3.85}$$

for 2-D flow, Therefore, by relating ω and ψ, we have

$$\omega = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)\,. \tag{3.86}$$

Or,

$$\nabla^2\psi = -\omega\,. \tag{3.87}$$

For steady flow, i.e. the flow quantities will independent of t, the vorticity equation can be simplified as

$$\nabla \times (\vec{\omega}\times\vec{u}) = \nu\nabla^2\vec{\omega}\,. \tag{3.88}$$

Because $\vec{\omega} = \omega\hat{z}$,

$$\vec{\omega}\times\vec{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ u & v & 0 \end{vmatrix} = (-\omega v)\hat{x} + (\omega u)\hat{y}\,, \tag{3.89}$$

therefore, the z component of the equation would become

$$\begin{aligned}
\nabla \times (\vec{\omega} \times \vec{u})_3 &= \left| \begin{matrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (-\omega u) & (\omega v) & 0 \end{matrix} \right|_3 \\
&= \frac{\partial}{\partial x}(\omega u) + \frac{\partial}{\partial y}(\omega v) \\
&= u\frac{\partial \omega}{\partial x} + u\frac{\partial \omega}{\partial y} + \omega\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) \\
&= \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} + \omega\left(\frac{\partial}{\partial x}\frac{\partial \omega}{\partial y} + \frac{\partial}{\partial y}\frac{-\partial \omega}{\partial x}\right) \\
&= \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}
\end{aligned} \tag{3.90}$$

Therefore, another relation between ψ and ω would be

$$\nabla^2 \omega = \frac{1}{\nu}\left(\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}\right). \tag{3.91}$$

And, from the Momentum equation, we can obtain the relation between P and ψ

$$\nabla^2 P = 2\rho\left[\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2} - \left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2\right]. \tag{3.92}$$

3.5 Flow Pass Rectangular Obstacle

3.5.1 Governing Equations

Instead of solving the equations relating ψ and ω immediately, we first non-dimensionalize the equation first. If we let w (width) to be length scale of the obstacle, u_0 to be the velocity scale of the inflow condition and $[f]$ to be the dimension of the quantity f, then

$$[\psi] = u_0 w \text{ and } [\omega] = u_0/w\,. \tag{3.93}$$

Therefore, the non-dimensionalized equations can be written as

$$\begin{aligned}
\nabla^2 \psi &= -\omega \\
\nabla^2 \omega &= Re\left(\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}\right),
\end{aligned} \tag{3.94}$$

with Re defined to be $u_0 w/\nu$.

This step would be important in which all the cases of the flow passing through the rectangular obstacle would be included here. No matter how you choose the strength of the inflow fluid or the size of the rectangle, or even the type of the fluid, Re would reflect all these combinations. As a result, by simply varying the value of Re, all situations can be solved numerically by considering one parameter, Re.

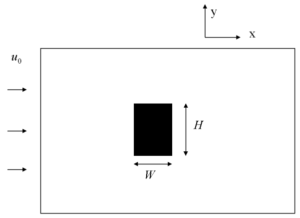

Figure 3.16: Computational Domain.

3.5.2 Numerical Methods

To solve the Poisson equation with the form of

$$\nabla^2 \phi(x, y) = f(x, y)\,, \tag{3.95}$$

we discretize the equation as

$$\phi^{n+1}_{i,j} = \phi^n_{i,j} + \frac{r}{4}\left(\phi^n_{i+1,j} + \phi^{n+1}_{i-1,j} + \phi^n_{i,j+1} + \phi^{n+1}_{i,j-1} - 4\phi^n_{i,j} - h^2 f^n_{i,j}\right). \tag{3.96}$$

By choosing different value of r, we have different methods: $1 < r \leq 2$ for over-relaxation; $r = 1$ for Gauss Seidal and $0 < r \leq 1$ for under-relaxation. For the problem of stability, we will generally set r to be in the under-relaxation range.

Therefore, the governing equations can be discretized as

$$\begin{aligned}
\psi^{n+1}_{i,j} &= \psi^n_{i,j} + \frac{r_\psi}{4}\left(\psi^n_{i+1,j} + \psi^{n+1}_{i-1,j} + \psi^n_{i,j+1} + \psi^{n+1}_{i,j-1} - 4\psi^n_{i,j} - h^2\omega^n_{i,j}\right) \\
\omega^{n+1}_{i,j} &= \omega^n_{i,j} + \frac{r_\omega}{4}\left(\omega^n_{i+1,j} + \omega^{n+1}_{i-1,j} + \omega^n_{i,j+1} + \omega^{n+1}_{i,j-1} - 4\omega^n_{i,j} - \frac{Re}{4}g^n_{i,j}\right)
\end{aligned} \tag{3.97}$$

with

$$g^n_{i,j} = (\psi^n_{i,j+1} - \psi^{n+1}_{i,j-1})(\omega^n_{i+1,j} - \psi^{n+1}_{i-1,j}) - (\psi^n_{i+1,j} - \psi^{n+1}_{i-1,j})(\omega^n_{i,j+1} - \psi^{n+1}_{i,j-1})\,. \tag{3.98}$$

Because this is a system of equations, to solve it numerically, we sub-iterate ψ and ω. Because the iterative scheme will easily become unstable, two things could be done. The first one is to reduce the relexation parameters r_ψ and r_ω. If the solution still do not converge at some Reynolds number Re_0, we could use the solution obtained at $Re_0 - \delta$ as the initial condition for that. For example, to

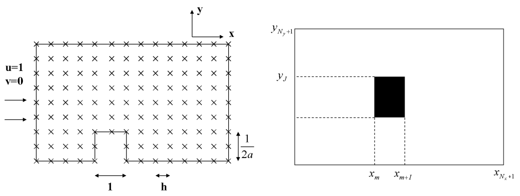

Figure 3.17: Domain Discretization.

solve the system at $Re = 100$, instead of setting all zeros as the initial condition, we use the solution obtained at $Re = 50$ as the initial guess. This will normally give a faster rate of convergence.

3.5.3 Boundary Conditions

Because the solution obtained would be symetric with respect to the x-axis, therefore, instead of computing all the solution in the domain, we will just iterate one half of the whole domain. This will increase the rate of convergence more than a half.

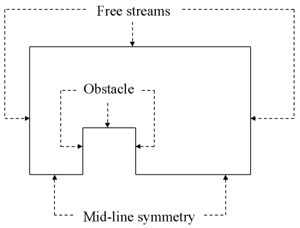

Figure 3.18: Boundary lines.

By doing this, the boundaries for this computational problem can be classified into three types: free stream boundaries, mid-line boundary and obstacle boundaries.

Free Stream

Free Streams include the upper boundary, inlet and the outlet. On the boundary, u would be equal to u_0 and this value would be set to be 1. v would be equal to 0. Therefore, because

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}, \tag{3.99}$$

we will have

$$\psi = y + c, \tag{3.100}$$

with some real number c. Because

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{3.101}$$

we will set ω to be 0 on these boundaries.

Another approach for the outlet is the normal derivatives boundary condition. If we use this type of boundary condition, we will let

$$\psi_{N_x+1,j} = \psi_{N_x,j} \text{ and } \omega_{N_x+1,j} = \omega_{N_x,j}, \tag{3.102}$$

such that $\psi_x = \omega_x = 0$ at the outlet.

Mid-line

Because we have set the x-axis to be the symmetry line, therefore, we will have

$$u(x,-y) = -u(x,y), \; v(x,-y) = -v(x,y) \text{ and } \omega(x,-y) = -\omega(x,y). \tag{3.103}$$

Therefore, at the mid-line, we will set $v = \omega = 0$. And, because

$$v = -\frac{\partial \psi}{\partial x}, \tag{3.104}$$

this implies ψ would be equal to a constant on the mid-line. For simplicity, this constant is set to be 0. In order to match the stream function on the mid-line to that in the free stream boundaries, we take the constant c in the above case to be also 0.

Obstacle

To find out the boundary conditions of the obstacle boundaries, we will use two conditions, the non-penetration condition and no-slip condition.

Non-penetration condition states that no fluid can pass this boundary and therefore, u would be 0 on the side (1) and (2). This means $\psi_y = 0$ on these sides. In order to match with the mid-line condition, $\psi = 0$ on side (1) and (2). For side (3), because $v = 0$ on this side, we will have $\psi_x = 0$. To match with side (1) and (2), ψ would also equal to zero here.

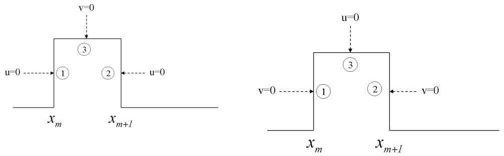

Figure 3.19: Non-penetration condition and no-slip condition.

No-slip condition states that the fluid will not slip on these sides. This means $v = 0$ on side (1) and (2) and $u = 0$ on side (3). Then, we will have

$$\psi_x = 0 \text{ and } \psi_y = 0\,. \tag{3.105}$$

They are the Normal derivatives boundary condition. To obtain the boundary conditions for the vorticity, we have to consider each side separately.

On side (1),

$$\omega = -\frac{\partial^2 \psi}{\partial x^2}\,. \tag{3.106}$$

By applying Taylor series expansion to the stream function ψ, we have

$$\psi(x_m - h, y) = \psi(x_m, y) - h\,\frac{\partial \psi}{\partial x}\bigg|_{(x_m,y)} + \frac{1}{2}h^2\,\frac{\partial^2 \psi}{\partial x^2}\bigg|_{(x_m,y)} + \cdots\,. \tag{3.107}$$

By using the non-penetration condition and no-slip condition, the first two terms after the equal sign would be equal to 0. And, this implies

$$\omega(x_m, y) = \frac{-2}{h^2}\psi(x_m - h, y)\,. \tag{3.108}$$

By using the same technique, we expanding the stream function next to side (2) as

$$\psi(x_{m+I}+h, y) = \psi(x_{m+I}, y)+h\,\frac{\partial \psi}{\partial x}\bigg|_{(x_{m+I},y)} + \frac{1}{2}h^2\,\frac{\partial^2 \psi}{\partial x^2}\bigg|_{(x_{m+I},y)} + \cdots\,, \tag{3.109}$$

and this implies

$$\omega(x_{m+I}, y) = \frac{-2}{h^2}\psi(x_{m+I} + h, y)\,. \tag{3.110}$$

It can be shown that

$$\omega(x, y_J) = \frac{-2}{h^2}\psi(x, y_J + h)\,, \tag{3.111}$$

on side (3).

Summary

Therefore, the boundary conditions can be summarized as in fig.3.19 and fig.3.20.

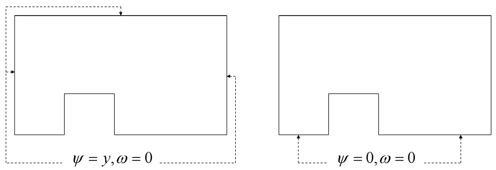

Figure 3.20: Boundary conditions.

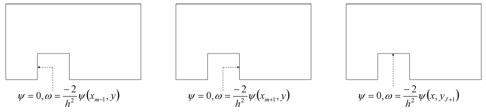

Figure 3.21: Boundary conditions.

3.5.4 Visualization

Of course, if there is any problem when you are using a Matlab command, you can simply type

>> **help** command

in the Matlab environment. Here, we mention two useful commands.

contour(x, y, z, v)

This command provides a contour plot of the 2×2 matrix z using vector x and y for axis scaling. If the contour levels are going to be decided manually, another vector v can also be put as a parameter.

fill(x, y, z)

When we are plotting the solution of the problem, we can fill a region in the Matlab figure with a color. The vectors x and y give the coordinates of the polygon. If the color of the region is decided to be black, $'k'$ would be used as an argument in the position of z.

3.5.5 Some Numerical Solutions

Fig.3.21 show the numerical solutions obtained for two different values of Re. The contour plot of the stream function, therefore the stream lines, are shown in the upper half of the solution. The lower half of the plot shows the contour plot of the vorticity, i.e. ω.

As we can see, when the value of Re is about 90, swirl will appear at the back of the obstacle. The fluid is divided into 2 part. The first part is the incoming fluids which moves to the right, while the other part is the wake which move circulating at the back of the obstacle, without any communication between them.

It can also be shown that larger the value of Re, longer and higher the wake.

Similar solutions can also be obtained in fig.3.22 when we use a different type of boundary conditions, normal derivative boundary conditions. Notice that the solutions at the outlet is smoothenned using this type of condition.

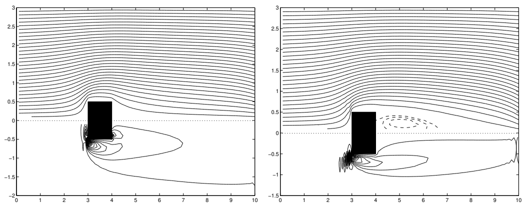

Figure 3.22: Solution obtained for Re equals (a) 30 and (b) 90.

3.6 Flow Passes Circular Obstacle

3.6.1 Governing Equations

If the rectangular obstacle is now replaced by a circular one, it would be better to use polar coordinates to to express the governing N-S equations. Before the

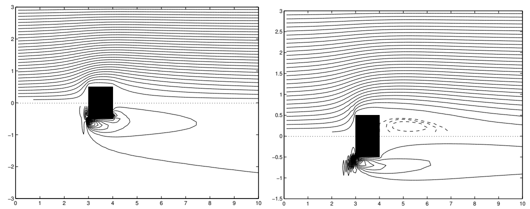

Figure 3.23: Solution obtained for Re equals (a) 30 and (b) 90 using normal derivatives boundary conditions.

transformation, we first take care of the units first. If the unit of the quantity f is denoted by $[f]$, we have

$$[u] = u_0 \text{ and } [x] = r_0 \,, \tag{3.112}$$

the Reynolds number is now defined to be

$$Re = \frac{2u_0 r_0}{\nu} \,. \tag{3.113}$$

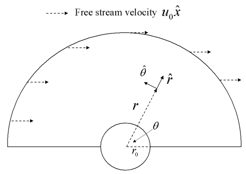

Figure 3.24: Flow Passes Circular Obstacle.

The relationship between the cartesian coordinates and polar coordinates are

$$x = r\cos\theta \text{ and } y = r\sin\theta \,, \tag{3.114}$$

or

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right) \,. \tag{3.115}$$

Therefore, from the relation

$$\hat{x} = \cos\theta\hat{r} - \sin\theta\hat{\theta} \text{ and } \hat{y} = \sin\theta\hat{r} + \cos\theta\hat{\theta}\,, \tag{3.116}$$

the free stream velocity can be written as

$$\vec{u} = u_0\hat{x} = u_0\cos\theta\hat{r} - u_0\sin\theta\hat{\theta} = u_r\hat{r} + u_\theta\hat{\theta}\,. \tag{3.117}$$

From the Continuity equation,

$$\begin{aligned} \nabla\cdot\vec{u} &= 0 \\ \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial\theta}u_\theta &= 0\,, \end{aligned} \tag{3.118}$$

we define the stream function $\psi = \psi(r,\theta,t)$ such that

$$\frac{\partial\psi}{\partial\theta} = ru_r \text{ and } -\frac{\partial\psi}{\partial r} = u_\theta\,. \tag{3.119}$$

And, the vorticity vector in cylindrical coordinates can be expressed as

$$\vec{\omega} = \nabla\times\vec{u} = \left[\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial}{\partial\theta}u_r\right]\hat{z}\,, \tag{3.120}$$

which means

$$\omega = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial}{\partial\theta}u_r\,. \tag{3.121}$$

Therefore, the governing equations relating the stream function and vorticity equation can be written as

$$\begin{aligned} \nabla^2\psi &= -\omega \\ \nabla^2\omega &= \frac{Re}{2}(\vec{u}\cdot\nabla)\omega = \frac{Re}{2}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\omega}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial\omega}{\partial\theta}\right)\,, \end{aligned} \tag{3.122}$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} = \frac{1}{r^2}\left[(r\frac{\partial}{\partial r})(r\frac{\partial}{\partial r}) + \frac{\partial^2}{\partial\theta^2}\right]\,. \tag{3.123}$$

In order to simplify the calculation, we introduce a modified polar coordinates by

$$r\frac{\partial}{\partial r} = \frac{\partial}{\partial\xi}\,. \tag{3.124}$$

Therefore,

$$\int_0^\xi d\xi' = \int_{r_0}^r \frac{1}{r'}dr'\,, \tag{3.125}$$

which implies

$$\xi = \ln\frac{r}{r_0} \text{ and } r = r_0e^\xi\,. \tag{3.126}$$

By doing this, the Laplacian can be much simplified to

$$\nabla^2 = e^{-2\xi}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2}\right)\,, \tag{3.127}$$

and the governing equations can be written as

$$\begin{aligned} \left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2}\right)\psi &= -e^{2\xi}\omega \\ \left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2}\right)\omega &= \frac{Re}{2}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\omega}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial\omega}{\partial\theta}\right)\,. \end{aligned} \tag{3.128}$$

3.6.2 Boundary Conditions

Same as in the previous section, the solution obtained would be expressed to be symmetric along x-axis. Therefore, the computational domain would be given by

$$\theta\in[0,\pi] \text{ and } \xi\in[0,\xi_m]\,, \tag{3.129}$$

for some ξ_m.

Instead of having 8 different boundaries in the rectangular obstacle case, there are only 4 boundaries in this coordinates representation. The side $\xi = 0$ corresponds to the surface of the cicular obstacle, the side $\xi = \xi_m$ represents the free stream boundary and the remaining two sides are the mid-line in front of and the back of the obstacle.

Free Stream

Because

$$\frac{\partial\psi}{\partial\theta} = ru_r = u_0r\cos\theta \text{ and } -\frac{\partial\psi}{\partial r} = u_\theta\,, \tag{3.130}$$

we have

$$\psi = u_0r\sin\theta = e^\xi\sin\theta\,, \tag{3.131}$$

by taking $u_0 = r_0 = 1$.

From the definition of ω, we have

$$\omega = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial r}(-ru_0\sin\theta) - \frac{1}{r}\frac{\partial}{\partial\theta}(u_0\cos\theta)\,, \tag{3.132}$$

which is equal to zero.

Mid-lines

The boundary conditions should be the same as in the previous case. This means $\psi = \omega = 0$ on both sides.

Obstacle Surface

From the non-penetration condition and no-slip condition, we have $u_r = u_\theta = 0$. This implies

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0 \text{ and } -\frac{\partial\psi}{\partial r} = 0\,. \tag{3.133}$$

In order to match with the mid-line condition, $\psi = 0$ on this boundary.

To find out the condition for ω, we apply Taylor's series expansion to $\psi_{1,j}$ about $\psi_{0,j}$ similar to the situation in the previous section, we have

$$\psi_{1,j} = \psi_{0,j} + h\left.\frac{\partial\psi}{\partial\xi}\right|_{0,j} + \frac{h^2}{2}\left.\frac{\partial^2\psi}{\partial\xi^2}\right|_{0,j} + \cdots\,. \tag{3.134}$$

By applying the non-penetration condition and no-slip condition to the first two terms in the expression after the equal sign, we can obtain

$$\omega_{0,j} = \frac{-2\psi_{1,j}}{h^2}\,. \tag{3.135}$$

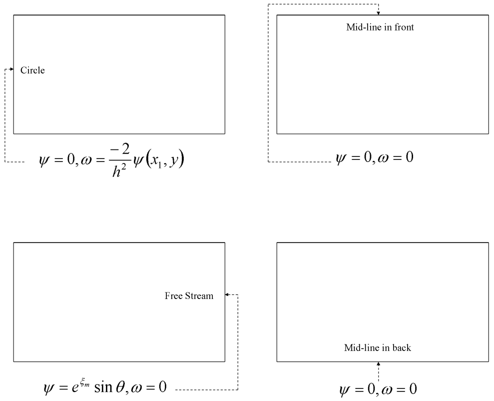

Figure 3.25: Boundary conditions.

Summary

Therefore, the boundary conditions can be expressed as in fig.3.25.

3.6.3 Interpolation Of The Solutions

Because the computational domain is in the $\xi - \theta$ space, we have to transform it back to the $x - y$ plane for visualization. Therefore, although the grid points are uniform in the computational plane, they would be uniform in the visualization plane anymore. A grid point in the $x - y$ plane may be corresponding to a position somewhere between 4 adjacent grids in the $\xi - \theta$ plane.

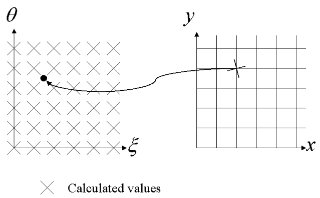

Figure 3.26: Reason for the interpolations.

In order to plot the solution in the $x - y$ plane uniformly, interpolations have to be made. This is a problem for 2 dimensional interpolation. Here states the basic idea for one dimensional interpolation:

For one dimensional, given any values of a known function $f(x)$ at points $x_1, \cdots, x_n$, we amy fit a polynomial to the data in order to find an arbitrary x such that $x_1 \leq x \leq x_n$.

If we are going to do a two dimensional interpolation at the point (x', y'), we first interpolate to find all values of $f(x', y_i)$. Then, we do another interpolation process to this n number of data to find out the value of $f(x', y')$.

3.6.4 Some Numerical Solutions

Here, we include two plot of the stream lines for this circular obstacle case when Re equals to 15. The upper half of fig.3.27a shows the contour plot of the streamfunction, i.e. the streamlines. The lower half of the diagram represents the contour plot of the vorticity. If we look closer to the circular obstacle, we can see that a circular wake appears at the back of the obstacle, as shown in fig.3.27b. The solutions obtained before interpolations are also plotted, as shown in fig.3.28.

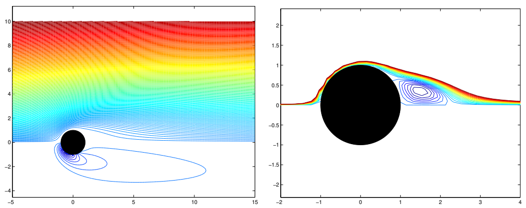

Figure 3.27: Streamlines and the values of the vorticity obtained at Re equals 15.

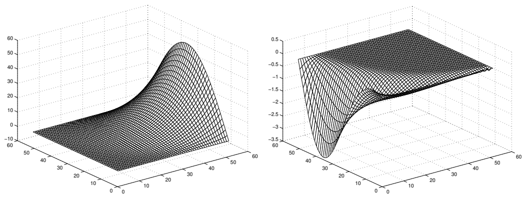

Figure 3.28: $\psi(\xi, \theta)$ and $\omega(\xi, \theta)$ obtained numerically.

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