- Pendulum equation

$$\frac{d^2\theta}{dt^2}+\frac{1}{q}\frac{d\theta}{dt}+\sin\theta = g\cos\omega t$$

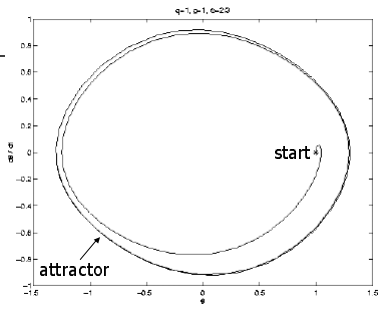
- with q=4, g=3/2 and w=2/3 for chaotic pendulum.

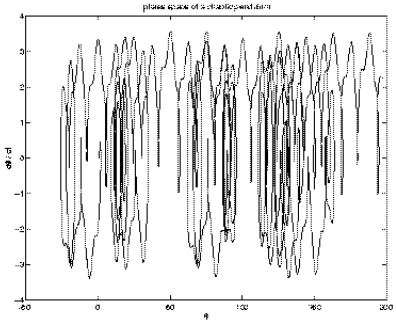
1. Phase Space

- Plot the solution in the form of

$$\frac{d\theta}{dt} \text{ vs } \theta(t)$$

- If the system is periodic, the curve in phase space will attracted to "a point" or "a closed curve". We call it an **attractor**.





But, for a chaotic pendulum, we can't see any attractor in the phase space.

2. Poincare Section

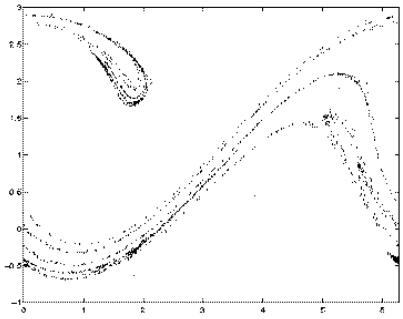
- There is no need to plot every solution we obtain:

$$\theta(t_i) \text{ and } \left.\frac{d\theta}{dt}\right|_{t=t_i} \quad \text{for } t_i = i\Delta t$$

- If the motion is period one (period of the motion= 2π / w, where w= frequency of the driving force), then we want only one point in the phase space.

- Therefore, in general, we want:
  - n points in the phase space represents a periodic motion with period= $2n\pi/w$
- That means, to obtain the Poincare Section, we plot those solution at $t=2n\pi/w$.

- So, for the chaotic pendulum we are studying, w=2/3. This means we plot the solution at $t=2n\pi/(2/3)=3n\pi$, for n=1,2,...
- The figure we obtain for the chaotic pendulum will sometime be called **Strange Attractor**.



- In order to obtain a good picture, there are several things we have to notice:
  - We drop first several periods before printing the data.
  - We have to map all those data to the range $(0, 2\pi) \times (-\pi, \pi)$ or $(-\pi, \pi) \times (-\pi, \pi)$.
  - At least few hundreds of points should be plotted.

- To plot a smaller point in the MATLAB environment, we use
  - >> set(gcf,'DefaultLineMarkerSize',#)
  - Where #=1,...,6. Smaller the number, smaller the point. (1: finest dot, 6: default)
- Accurate solution is required. Better to use double precision instead of single precision.

Technical point 1:

- How to check whether the time is $t=2\pi n/w$ (if w=2/3, then $t=3n\pi$) or not?
- Problem: If you use the command like:
  - if (mod(t,3 π).eq.0) then...
- The program works or not will then depends on the value dt you choice.

- M1: $dt=\pi/m$, where m=10, say.
- M2: not to use the previous command mod(), but instead, the programmer calculate the number of step required to meet the particular time. If $dt= \pi/10$, the you printout the solution every 30 time step.

Technical point 2:

- How to represent π ?
- M1: π=3.141592654
- M2: π=4*atan(1.)

3. Spectrum Analysis

- Instead of studying the solution in the phase space, we will study it in the **frequency space** by using the algorithm of FFT (Fast Fourier Transform).

- Solution we obtained before: $f(t)$, is the solution in the real space, as a function of real time.
- Transformed solution: $\hat{f}(\omega)$, is the solution in the frequency space, as a function of ω , the wave frequency.

- By expanding the function $f(t)$ by Fourier Transform, we get

$$f(t)=\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}d\omega \text{ and } \hat{f}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\omega)e^{-i\omega t}dt$$

- Remind you that $\hat{f}(\omega)$ is the same solution of $f(t)$, but represented in the frequency space.

- If the frequencies are discretized, integration will be replaced by summation:

$$f=\sum_{n=-\infty}^{\infty}\hat{f}_n e^{i\omega_n t}d\omega$$

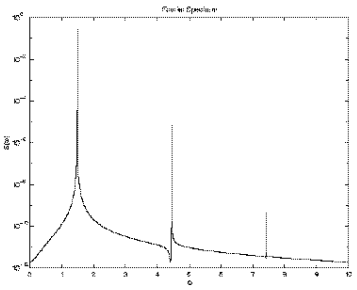
- But in computation, we can't duel with infinity. Therefore, the high frequency modes are truncated:

$$f=\sum_{n=-\max}^{\max}\hat{f}_n e^{i\omega_n t}d\omega$$

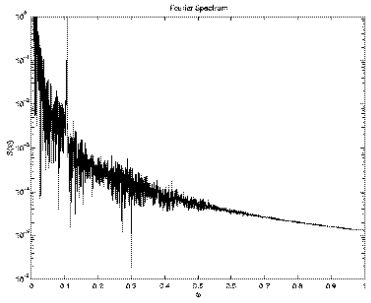
- The Fourier Spectrum function S(w) is defined as

$$S(\omega)=\hat{f}(\omega)\hat{f}(\omega)^*=\left\|\hat{f}(\omega)\right\|^2$$

- Then, the spectrum is the figure of S(w) vs. w of the motion.



Spectrum for the pendulum with q=1e10, g=0, dt=0.1 and N=2^18.



Spectrum for the pendulum with q=4, g=3/2, w=2/3, dt=0.01 and N=2^18.

- Broadband power spectrum is an indication of chaos. The reason is this:
  - Each peak in the spectrum represents that particular frequency contributes to the solution in the real space.
  - More the peak, more interactions between more frequencies, and this makes the solution in the real space oscillates more violently.

- To obtain a good figure, we have to remember the following things:
  - At least several periods, for periodic motion, have to be included in the input data, $f(t_i)$.
  - First several periods will be dropped because the motion is still not steady.

- The program for FFT using MATLAB will be provided to you:
  - public_html/tutorial5/spectrum.m
  - The file u.dat should provide the time series data for the pendulum.
  - n represents the number of data, should be in the form of n=2^p, where p=17 or 18 in general.

- dt represents delta t you used in the program.
- The data in the sample $u(t)$ should be evenly spread, i.e.

$$t_i = i\Delta t, \text{for constant } \Delta t$$