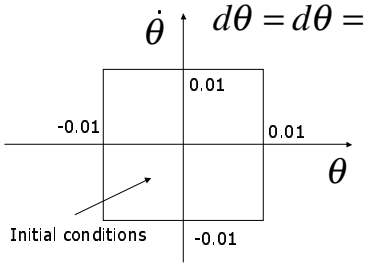
Sensitive to Initial Condition

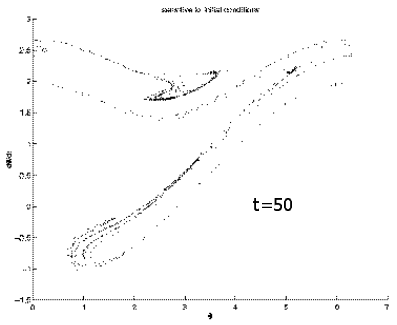
A Suggested Solution to
Question 1 in Tutorial 5

- One way to see the property of sensitive to initial condition of the chaotic pendulum system, we can see where is the pendulum in the phase space after a fixed time for a number of initial conditions:

So, there are more than 1600 points in the phase space as the initial conditions

$$d\theta = d\dot{\theta} = 0.0005$$





- One property of chaotic systems:
  - **Distance between any two initial conditions in the phase space can be arbitrary after sufficient time for a chaotic system.**

Bifurcation of Logistic Map

Math 333 Tutorial 7
24-3-2000

- Logistic Map can be used to model the population of an organism with x represents the frequency of a species

$$x_{n+1} = \mu x_n (1 - x_n)$$

for some parameter $0 < \mu < 4.$

- This parameter μ would depends on the living environment of the species, the supply of food or the life time of the organism.
- Giving the frequency of the species for the generation n, we can obtain the frequency of it in the next generation

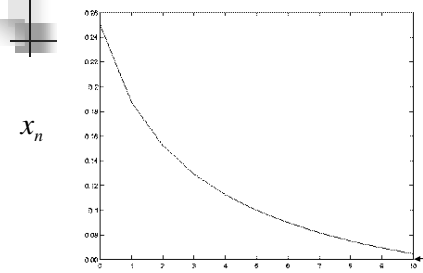
- For example, if

$$\mu = 1.0 \text{ and } x_0 = 0.25$$

- Then, the frequency of the next generation can be calculated to be

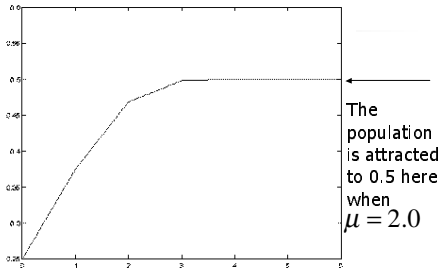
$$x_1 = 1 \times 0.25 \times (1 - 0.25) = 0.1875$$

$\mu = 1.0$



We say the population is attracted to the fixed point $x_n = 0$

$\mu = 2.0$



The population is attracted to 0.5 here when $\mu = 2.0$

- Both of the above motions are called Period 1 motion, because there are only 1 attraction point.
- From the analysis done in the lecture time by Dr. Chasnov, we can see that Period 1 motion can be found when

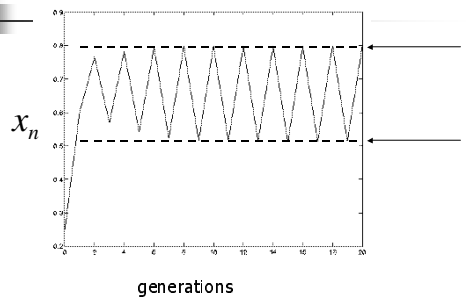
$$0 < \mu < 3$$

- Now, if we increase the value of μ to 3.2, we expected to obtain a Period 2 motion.
- That means

$$x_{n+2} = x_n$$

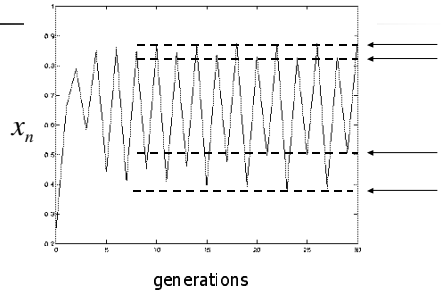
if n is large enough

$\mu = 3.2$



- This is similar to the analysis of the pendulum motion where the first several periods will not be considered, because the motion is still not "steady".
- When we further increase the value of μ, we see period of the motion doubled. We call this phenomena "Period doubling"

$\mu = 3.5$



Bifurcation Diagram

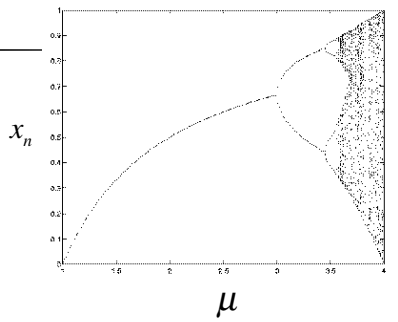
- Now, we collect all those period points for corresponding μ

μ	x	Period
1.0	0.0	1
2.0	0.5	1
3.2	0.799455, 0.513045	2
3.5	0.382815, 0.826937, 0.500893, 0.874997	4

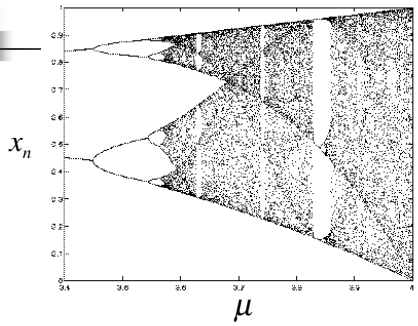
- If we re-fine the mesh by decreasing the difference between 2 values of μ and plot the **points** on the plane

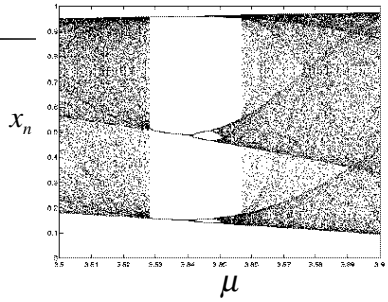
x_n VS μ

- What we get is the bifurcation diagram of the Logistic Map



- Remember that we have mentioned that the period of the motion is doubling. This means the period of the periodic points will increase from 1 to 2 to 4 to 8…
- But if we look closer to the range near 3.8, we will see a period 3 motion appears suddenly, from nowhere





- Li, T.Y and Yorke, J.A., *Period Three Implies Chaos.* American Mathematical Monthly, 82, 985-992
- The idea of a theorem mentioned in the paper is that:
  - **If period 3 motion is found in a dynamical system, a motion for any period also exists.**

Technical Point 1:

- Things we have to do in order to get a pretty picture:
  - 1. Remember not to use a large number of initial points, which represents the non-steady motion of the dynamics.
  - 2. Difference between two μ has to be small in order to eliminate the gaps in the diagram.

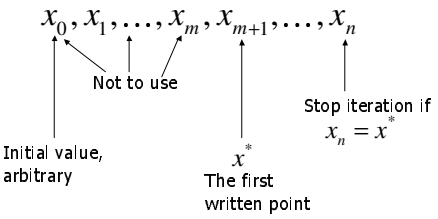
Technical Point 2:

- How do I know the number of points to be written out for a particular μ ?
- For example, how do I know I have to write 4 points when $\mu = 3.5$?
  - The answer for both of the questions is: No, you don't have to worry about that, let the computer do it for us.

- Method 1:
  - No matter what the value of μ is, write m points, say 50, to the data file. This mean we won't consider whether it is a motion with period 50 or not, we still write 50 points.
  - So, 50 points for period 1 motion; 50 points for period 2 motion; 50 points for period 64 motion…

- Problem: we can expect, the data file would be huge, in which there are so many redundant information where we don't need.
- There is no reason why we use 50 points to represent a period 1 motion. In this case, 49 pairs of data are redundant.

- Method 2:
  - Let the computer remembers the first written point x^*, then compare this value every time before writing another point to the data file. If a later value x_n is equal to x^*, stop the iteration and do another μ



- Something more:
  - 1. Remember that there is μ_∞ such that there it is "period infinity"? So, we have to fix the maximum number of points to be written out for a particular μ
  - 2. x_n Will never equal to x^* because of the error. So, what we check really is whether

$$\left\| x^* - x_n \right\| < eps$$

If it is so, we treat they are equal