The Hong Kong University of Science & Technology
Department of Mathematics
Introduction to Scientific Computation I
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A STUDY IN HÉNON MAP

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Abstract. *Hénon map is a two dimensional map suggested by the French astronomer Michal Hénon in 1976. It is a simplified model for the dynamics of the Lorenz system. The map creates a boomerang-like shaped attractor which is known as the Hénon attractor. The behaviour of the map corresponding to the control of parameters is studied and the results obtained has been analyzed.*

1. Introduction

The Hénon map consists of two variables and two parameters. It is a two dimensional map on the phase space (x, y). The choice of parameters will in fact promote to a quite different solution of the system. The system will become chaos with particular values of parameters and the chaotic behaviour of the system is the most interesting part of study. In this paper, I am going to dicuss the principle of the mapping, leading to its chaotic behaviour (the strange attractor), and the characteristics invovled. After these dicussions, conclusions will be driven at the end.

2. Governing equations

The Hénon map is defined by the transformation:

$$H(x, y) = (1 - ax^2 + y, bx) \tag{1}$$

or the iteration:

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + y_n \\ y_{n+1} &= bx_n \end{aligned} \tag{2}$$

in a two-dimensional (x,y) phase space, where a and b are parameters.
By choosing b=0, the system becomes a one dimensional quadratic map $x_{n+1} = 1 - a{x_n}^2$. When considering this dynamic system, b is related to the damping or dissipation in the system, and a is corresponding to the forcing or stress.

3. Properties of map

3.1. Transformation

The Hénon map can be separated into three maps:

(1)	$H_1(x, y) = (x, 1 - ax^2 + y)$	a nonlinear bending in the y-coordinate
(2)	$H_2(x, y) = (bx, y)$	a contraction in x-direction with contraction factor b
(3)	$H_3(x, y) = (y, x)$	a reflection at the diagonal

i.e. $H(x, y) = H_3(H_2(H_1(x, y)))$

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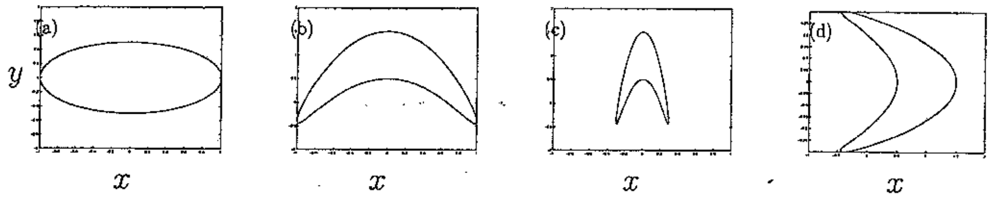

FIGURE 1. *Hénon transformation on an ellipse with parameters $a = 1.0$ and $b = 0.5$*

In the figure, (b) shows the bending H_1, (c) shows the contraction H_2 and (d) shows the reflection H_3. The ellipse is finally transformed to a horseshoe-like shape.

3.2. Area Reduction

By choosing $|b| < 1$, an original area after iterations will shink and be reduced by a factor of b by each iteration. Fig 1 provides the same phenomenon with $b = 0.5$.
The reduction factor is given by the absolute value of the determinant of the Jacobian matrix:

$$|\det DT(x,y)| = \left|\det \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}\right| = |b|.$$

Therefore, the smaller value of $|b|$, the increase in reduction of area. The value $|b| = 1$ will lead to conservation of the area.

3.3. Invertibility

Hénon map is an invertible map with $H^{-1}(x,y) = H_1^{-1}(H_2^{-1}(H_3^{-1}(x,y)))$ where H_1, H_2, H_3 are all invertible. It can also be written as a backward iteration :

$$\begin{aligned} x_n &= \frac{y_{n+1}}{b} \\ y_n &= x_{n+1} + \frac{a}{b^2} y_{n+1}^2 - 1 \end{aligned}$$

4. Fixed point

Fixed points (x_*, y_*) of the map exit when continuous iterations give the same point in phase space. Thus the fixed points will satisfy the following equations:

$$\begin{aligned} x_* &= 1 - a x_*^2 + y_* \\ y_* &= b x_* \end{aligned}$$

On solving, the fixed points of the henon map are:

$$\begin{aligned} x_{1,2} &= \frac{b - 1 \pm \sqrt{(b-1)^2 + 4a}}{2a} \\ y_{1,2} &= b x_{1,2} \end{aligned}$$

For $a < -\frac{(b-1)^2}{4} = -0.1225$, there is no fixed point exit since the roots are not realand when a grows beyond -0.1225, there exists two fixed points (x_1, y_1) and (x_2, y_2), where the former is attracting. The stability of the fixed point can be determined by the value $|\frac{x_{n+1} - x_*}{x_n - x_*}|$, the fixed point will be stable if the value is less than 1 as it indicates that the initial condition approachs the fixed point for each iteration.

5. Bifurcation

At particular a values, there will be an increase in number of fixed points, which gives rise to a period doubling. The system will undergo this continuously and bifurcate to a period of infinity, which is a chaotic system.

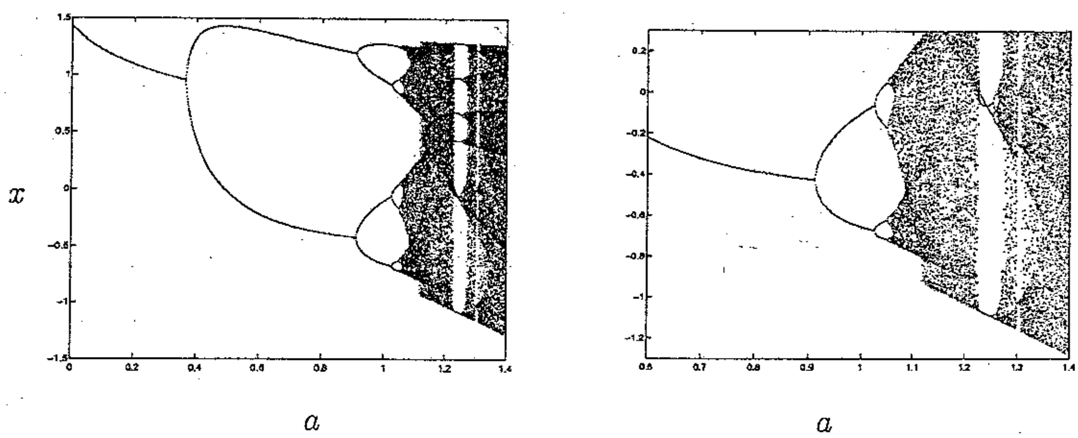

FIGURE 2. *Bifurcation diagram with parameter* $b = 0.3$ *and range of parameter* $0 < a < 1.4$

TABLE 1
The period doubling Cascade

k	Period	Parameter a_k	δ
0	1	-0.1225	
1	2	0.3675	
2	4	0.9125	4.844
3	8	1.026	4.3269
4	16	1.051	4.696
5	32	1.056536	4.636
6	64	1.05773083	4.7748
7	128	1.0579808931	4.6696
8	256	1.05803445215	4.6691
9	512	1.05804592304	4.6691
10	1024	1.05804837980	4.6694
11	2048	1.058048905931	

The above table shows the the parameter a_k for the period doubling cascade in Hénon transformation for $b = 0.3$. $a = 0.3675$ is the beginning of a period doubling transistion to chaos with $a_\infty \simeq 1.058048$. δ_k is defined to be equal to $\dfrac{a_k - a_{k-1}}{a_{k+1} - a_k}$. The Fergenbaum number $\delta = \lim_{n\to\infty} \delta_n = 4.669201609......$ which is an universal constant.

6. Periodic and chaotic behaviour

The parameters are controls of the system. The choice of these parameters can result (x,y) to iterate in a periodic or chaotic way.

Periodic case

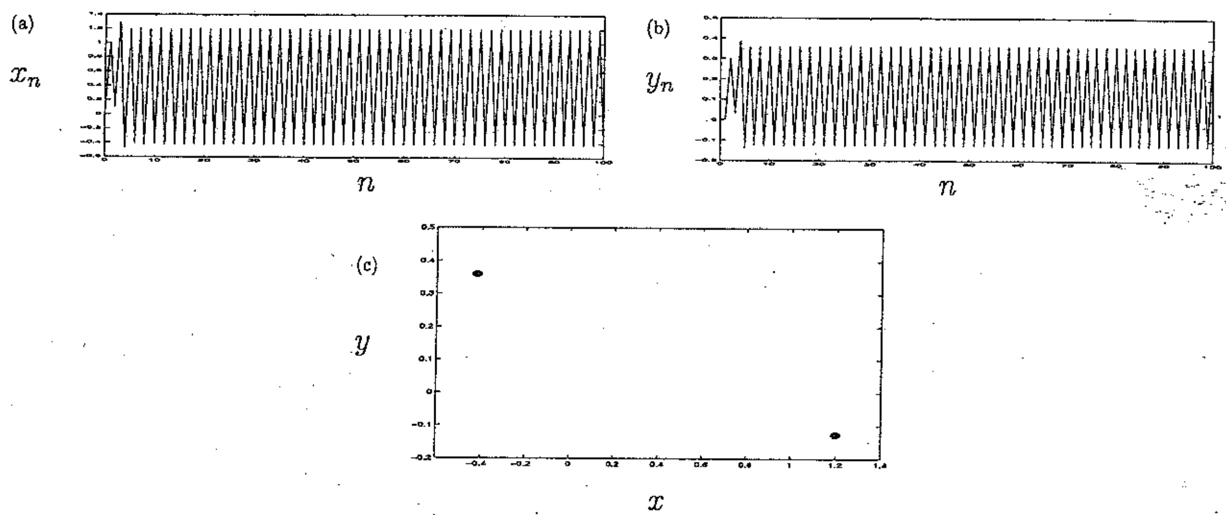

FIGURE 3. *(a) The time series of x, (b)The time series of y, (c) The period 2 attractor of Hénon map with parameters $a = 0.9$ and $b = 0.3$*

Chaotic case

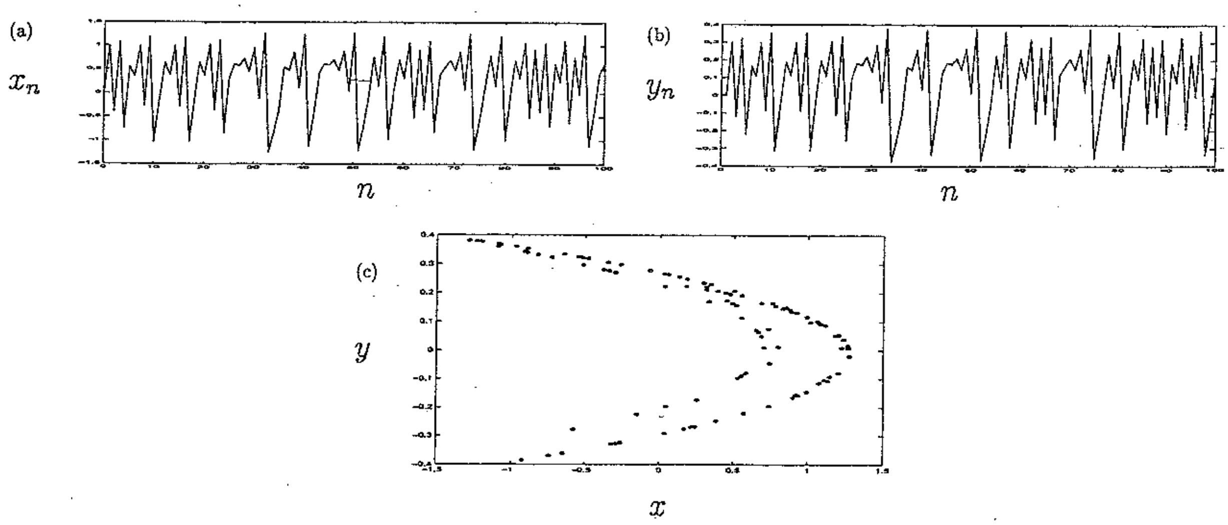

FIGURE 4. *(a) The time seies of x, (b) The time series oy y, (c) The strange attractor of Hénon map with parameters $a = 1.4$ and $b = 0.3$*

For different values of a, there exits different fixed points. For chaos, it does not behave in a periodic way and thus the number of fixed points increases to infinity and a strange attractor is produced.

7. Study in chaotic system

7.1. Strange attractor

The strange attractor indicates the long term behaviour in phase space of the chaotic system. The Hénon attractor is boomerang-liked and is self similar.

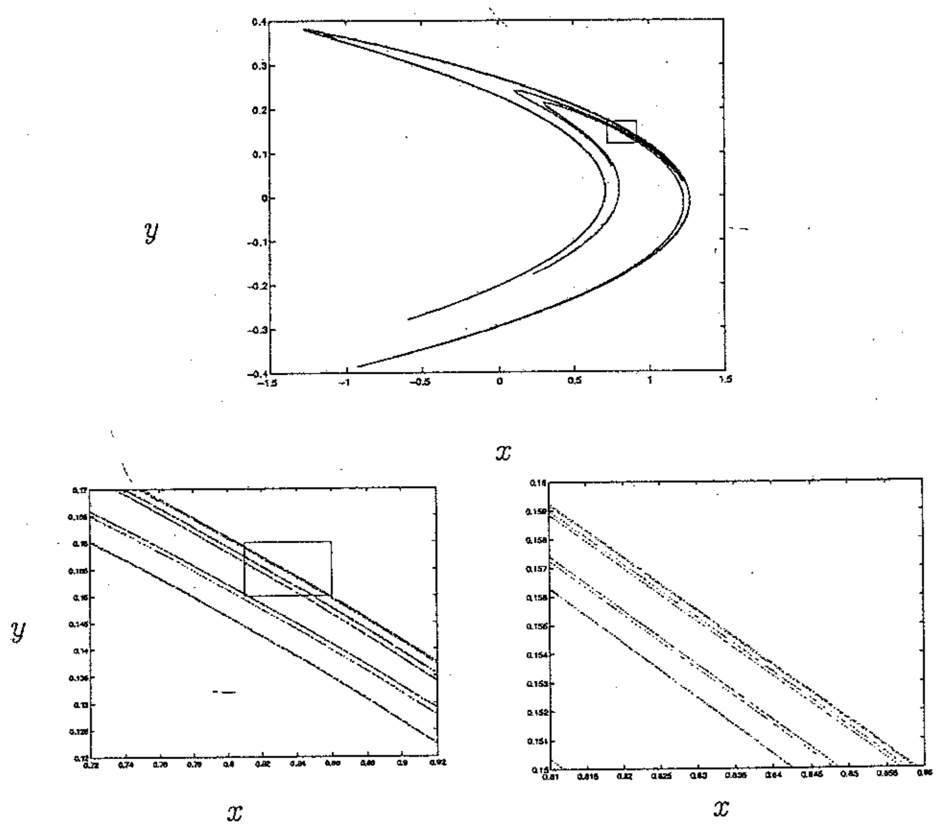

FIGURE 5. *The atrange attractor of Hénon map with parameters* $a = 1.4$ *and* $b = 0.3$

At $a = 1.4$, solutions seem to jump around the attractor randomly. It has the characteristic of a "fractal", which has a similar structure when it is scaled down.

7.2. Correlation dimension

As the attractor has a fractal structure, its fractal dimension can be calculated. One of the definition of fractal dimension is the correlation dimension. In calculating this correlation dimension, we consider the correlation between points in a ball of radius ϵ which scales as ϵ

raised to some power. Here we define:

$$\begin{aligned} d_{corr} &= \lim_{\epsilon \to 0} \frac{\log C(\epsilon)}{\log \epsilon} \\ C(\epsilon) &= \lim_{N \to \infty} \frac{1}{N^2} \sum_{i,j} \theta(\epsilon - ||x_i - x_j||) \end{aligned} \quad \text{where} \quad \theta(\alpha) = \begin{cases} 0 & \text{for } \alpha < 0 \\ 1 & \text{for } \alpha > 0 \end{cases}$$

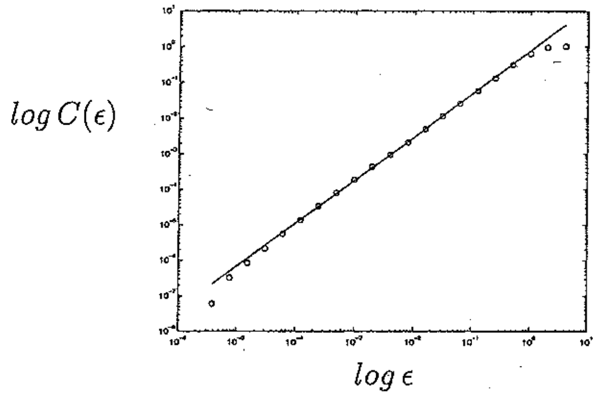

FIGURE 6. *The plot of* log $C(\epsilon)$ *vs* log ϵ

The figure is the log/log plot in which the slope is the correlation dimension. The correlation dimension is calculated to be 1.2090 ± 0.0064.

7.3. Trapping region

There is a trapping region in which no orbit with initial points in it can escape out of the region. Since the attractor attracts points near by, the attractor will be included in the region.

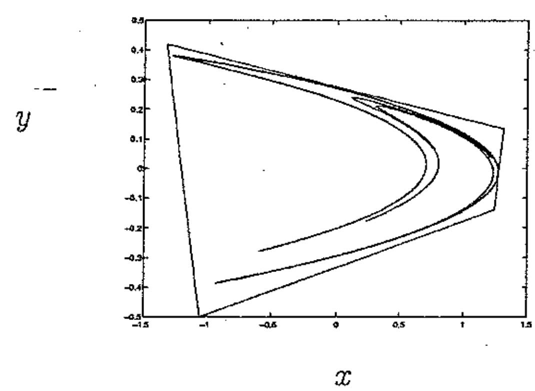

FIGURE 7. *The trapping region the chaotic system with parameters* $a = 1.4$ *and* $b = 0.3$

7.4. Sensitivity to initial condition

One of the important characteristics of the chaotic system is its sensitivity to initial conditions. A slight difference in initial data will lead to quite different behaviour. The time series of two different initial conditions are shown below and are compared to each other.

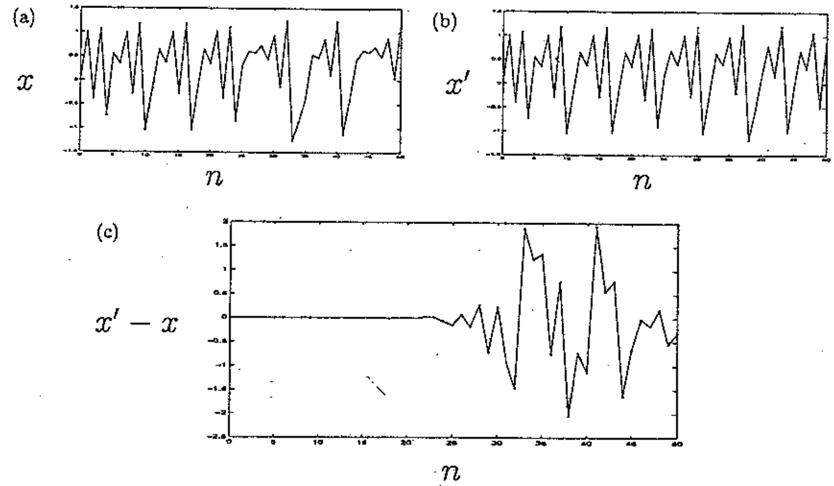

FIGURE 8. *(a) The time series of x with initial condition $(x_0, y_0) = (0, 0)$, (b) The time series of x' with initial condition $(x'_0, y'_0) = (0.00001, 0)$, (c) The difference between two time series of different initial conditions*

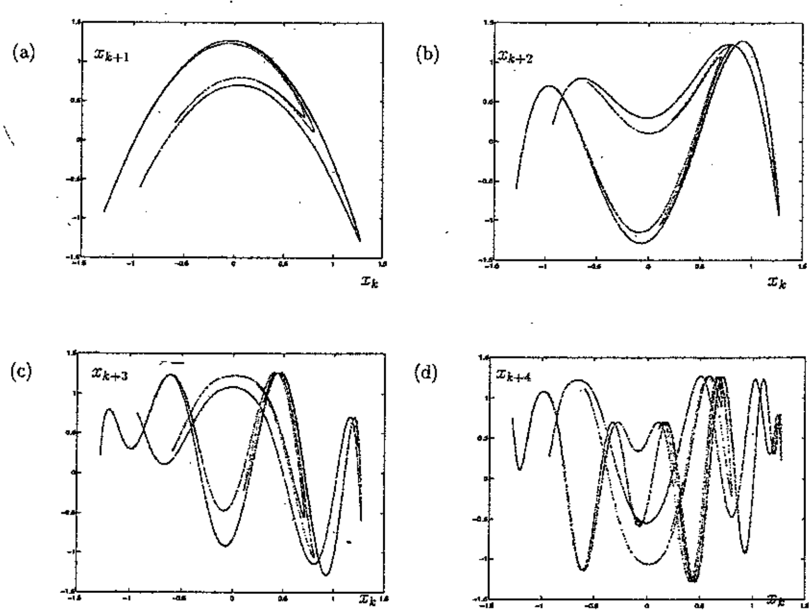

FIGURE 9. *Plot of x_{k+m} vs. x_k with (a) $m = 1$, (b) $m = 2$, (c) $m = 3$ (d) $m = 4$*

7.5. Correlation between ordits

Although the different initial conditions will finally turn to the attractor, there is no correlationship between each other even they are close together. Except for the case that the second initial point is a point in the first obit.

Since the initial condition (x_0', y_0') is set to be (x_m, y_m) which is in the orbit of initial condition (x_0, y_0), there is correlation between two orbits as shown in the figure.

8. Conclusion

The long term behaviour of this dynamic system of some chosen a would run into a simple pattern of motion which is the strange attractor, the study of this system gives a understanding of "chaos" which actually is not so random and arbitary as we have thought. Although this is not totally random in pratice, there is still a long way to go to completely understand its behaviour.

REFERENCES

[1] Addison P.S. 1997 . *Fractals and chaos: An illustrated course*, Institute of Physics Publishing Bristol and Philadelphia.
[2] Kim J. H., & Stringer J, (eds) 1992. *Applied chaos* , John Wiley & Sons, Inc.
[3] Peitgen H. O., Jürgens H.,& Saupe D. 1992. *Chaos and Fractals: New Frontiers of Science*, Springer-Verlag New York, Inc.
[4] Rasband S. N. 1990, *Chaotic dynamics of nonlinear systems*, John Wiley & Sons, Inc.