MATH333: Tutorial 10 (14-4-00)

1 Numerical Methods for Poisson Equation

Here in this tutorial, we will discuss how to solve a Poisson equation,

$$\nabla^2 u(x,y) = f(x,y)\,,$$

or, Laplace equation when $f(x,y)=0$.

To approximate the second derivatives, we use central difference to obtain smaller error,

$$\begin{aligned}
\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=x_i,y=y_j} &= \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h_x^2}\,,\\
\left.\frac{\partial^2 u}{\partial y^2}\right|_{x=x_i,y=y_j} &= \frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h_y^2}\,.
\end{aligned}$$

In general, we will use same $h=h_x=h_y$ for simplicity. Therefore, the problem can be discretized to

$$\frac{1}{h^2}\left\{\begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array}\right\} u_{i,j} = f_{i,j}\,.$$

To solve this system, there are several methods.

1.1 Direct Method

Therefore, we can obtain the matrix

$$Ax=b\,,$$

where

$$A=\begin{pmatrix}
-4 & 1 & 0 & \cdots & & 1 & 0 & \cdots & \\
1 & -4 & 1 & 0 & \cdots & & 1 & 0 & \cdots \\
0 & 1 & -4 & 1 & 0 & \cdots & & 1 & 0\cdots \\
\vdots & & & & & & & & \\
1 & 0 & \cdots & & & & & & \\
0 & 1 & 0 & \cdots & & & & & \\
\vdots & & & & & & & &
\end{pmatrix}, x=\begin{pmatrix} u_{11}\\ u_{12}\\ \vdots\\ u_{1m}\\ u_{21}\\ \vdots\\ u_{nm}\end{pmatrix},$$

and

$$b=(f_{11}f_{12}\cdots f_{1m}f_{21}\cdots f_{nm})^T+\vec{\epsilon},$$

with $\vec{\epsilon}$ relates to the boundary conditions.

This system can be solved by computing the inverse of A and then multiply it by the vector b. However, if you discretize the domain by 50×50 points, the size of this matrix A will already by 2500×2500. Remember the operation counts we did last semester? The number of operations needed will be in order of $O(2500^3)$, and which will take you several hours to do the inverse problem. This method is therefore not practical and will not be used in general.

1.2 Iterative Method

If there are a large number of points we want to compute in the domain, we will use iteration to solve it. Three iterative methods were introduced last semester in MATH 231, Gauss-Seidel, Jacobi and SOR. We will use SOR normally bcause of its speed of convergence.

The method of SOR iteration can be written as

$$u_{i,j}^{(k+1)} = u_{i,j}^{(k)} + \frac{\omega}{4}\left(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k+1)} - 4u_{i,j}^{(k)} - h^2 f_{i,j}\right) .$$

Using arbitrary initial condition you like, we perform the iteration until

$$\max_{i,j}\left|u_{i,j}^{(k+1)} - u_{i,j}^{(k)}\right| \leq \epsilon .$$

2 Examples

Remember that there will be no more exercise given in the tutorial section. The below questions are just some examples given to you. Hope you will spend some time in the holiday to really think about them. All the skills you used here will be very useful in your Paper II.

1. Using SOR iteration to solve the equation

$$\nabla^2 u(x,y) = xy(x-1)(y-1) ,$$

   on the region $0 \leq x \leq 1, 0 \leq y \leq 1$, with $u(x,y) = 0$ on all boundary except for $x = 0$, where $u = 1.0$.

2. Remember that there is an external parameter ω used in the SOR iteration. This parameter ω will largely affact the efficiency of the iteration method. By varying this parameter, observe how the number of iterations required varies.

3. Now, the boundary condition on the side $x = 0$ is changed to $u_x = 1$. By approximating the derivative by

$$\left.\frac{\partial u}{\partial x}\right|_{0,j} = \frac{-u_{2,j} + 4u_{1,j} - 3u_{0,j}}{2h} ,$$

   solve the Poisson equation.