First name:____________________ **Last name:**____________________

Student number:____________________ **Signature:**____________________

UNIVERSITY OF CALIFORNIA, IRVINE
DEPARTMENT OF MATHEMATICS

March 19, 2008

MATH3A (Introduction to Linear Algebra)
Instructor: Shingyu Leung

Duration: 110 minutes

No aids allowed

This examination paper consists of **7** pages and **8** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth. You can write on both sides of the paper.

Answer all questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (16)	
2 (15)	
Subtotal	

	Score
3 (8)	
4 (9)	
5 (9)	
Subtotal	

	Score
6 (15)	
7 (13)	
8 (15)	
Subtotal	

Total (100)	

1. [16] True or false. In the case of a true statement, just write down **TRUE/T**. In the case of a false statement, write down **FALSE/F AND** also disprove the statement or give an example to show that it is not always true.

   (a) If $\dim(V) = n$, then any set of $n - 1$ vectors in V must be linearly independent.

   (b) If S, T, and U are subspaces of an inner product space and $S \perp T$, $T = U^\perp$, then $S = U$.

   (c) If a set S of vectors in V contains the zero vector, then S is linearly dependent.

   (d) If $\dim(V) = n$, then there exists a set of $n + 1$ vectors in V that spans V.

   (e) If S, T, and U are subspaces of a vector space and $S \perp T$, $T \perp U$, then $S = U$.

2. [15] Let

$$\mathcal{F} = \{1, 1 - x, 1 + x^2\}\,. \tag{1}$$

(a) Show that $\mathcal{F}$ forms a basis for P_3.

(b) Find the transition matrix $\mathbf{B}$ representing the change of coordinates on P_3 from $\mathcal{F}$ to the ordered basis $\{1, x, x^2\}$.

(c) Using (b), or otherwise, find the coordinates of $p(x) = a_0 + a_1 x + a_2 x^2$ with respect to the the basis $\mathcal{F}$.

3. [8] Let $A \in \mathbb{R}^{319\times 2008}$ with rank equal to 200. Write down the dimensions of the fundamental subspaces $R(A)$, $R(A^T)$, $N(A)$ and $N(A^T)$.

4. [9] Let $L : V \to V$ is a linear transformation and $\mathbf{x} \in \ker(L)$. Show that $L(\mathbf{v}-2008\mathbf{x}) = L(\mathbf{v})$ for all $\mathbf{v} \in V$.

5. [9] Let S be a subspace of an inner product space spanned by $\mathbf{u}_1, \cdots, \mathbf{u}_{319}$. Show that $\mathbf{v} \in S^{\perp}$ implies $< \mathbf{v}, \mathbf{u}_k >= 0$ for $k = 1, \cdots, 319$.

6. [15] Consider $C[-\pi, \pi]$ with the inner product

$$< f, g >= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx\,, \tag{2}$$

and the norm $||f||^2 =< f, f >$.

(a) Show that the set

$$\mathcal{F} = \left\{ \frac{1}{\sqrt{2}}, \cos x, \sin 2008x \right\} \tag{3}$$

is an orthonormal set of vectors.

(b) Determine the value of

$$\left\| \frac{1}{\sqrt{2}} + \cos x + \sin 2008x \right\| \,. \tag{4}$$

(c) Determine the projection of $h(x) = \sin x$ onto the subspace of $C[-\pi, \pi]$ spanned by $\mathcal{F}$.

7. [13] Let

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 2 \\ 3 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

(a) Use the Gram-Schmidt process to obtain an orthonormal basis for $R(\mathbf{A})$ with respect to the usual scalar product defined in $\mathbb{R}^4$.

(b) Determine an orthogonal matrix $\mathbf{Q}$ and an upper triangular matrix $\mathbf{R}$ such that $\mathbf{A} = \mathbf{QR}$.

8. [15] Let $L : \text{Span}(1, \sin x, \cos x) \to P_3$ defined by

$$L(f(x)) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \,. \tag{6}$$

For example,

$$L(\cos x) = \cos(0) - \sin(0)x - \frac{\cos(0)}{2}x^2 = 1 - \frac{1}{2}x^2 \in P_3 \,. \tag{7}$$

(a) Show that L is a linear operator.

(b) Find the matrix representation of L with respect to the ordered bases $\mathcal{E} = \{1, \sin x, \cos x\}$ and $\mathcal{F} = \{1, 1+x, 1+x-x^2\}$.

(c) Determine the kernel and the range of L.

End of examination
Total pages: 7
Total marks: 100