First name: ____________________ **Last name:** ____________________

Student number: ____________________ **Signature:** ____________________

UNIVERSITY OF CALIFORNIA, IRVINE
DEPARTMENT OF MATHEMATICS

February 8, 2008

MATH3A (Introduction to Linear Algebra)
Instructor: Shingyu Leung

Duration: 50 minutes

No aids allowed

This examination paper consists of **5** pages and **6** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth. You can write on both sides of the paper.

Answer all questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (20)	
2 (15)	
Subtotal	

	Score
3 (20)	
4 (15)	
Subtotal	

	Score
5 (15)	
6 (15)	
Subtotal	

Total (100)	

1. [20] True or false. Just write the word **TRUE/T** or **FALSE/F**. No explanation is necessary.

   (a) Let $\mathbf{x}_1 = (1,0,1)^T$, $\mathbf{x}_2 = (1,1,3)^T$ and $\mathbf{x}_3 = (2,1,4)^T$. The set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ spans $\mathbb{R}^3$.

   (b) Let $A, B \in \mathbb{R}^{n\times n}$ such that $AB = \mathbf{0}$. Then rank(A)+rank$(B) \leq n$.

   (c) Let $f_1(x) = e^{ax}$ and $f_2(x) = e^{bx}$. f_1 and f_2 are linearly independent if $a \neq b$.

   (d) If the set of vectors $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is a basis for a vector space V, then $\mathbf{u}_1, \cdots, \mathbf{u}_n, \mathbf{u}_{n+1}$ are linearly dependent.

   (e) If $\dim(V) = n$, then there exists a set of $n+1$ vectors in V that spans V.

   (f) Let $A \in \mathbb{R}^{m\times n}$. If $\dim(N(A)) = 0$, the column vectors of A are linearly independent.

   (g) The set of all vectors of the form $(a, a+b, a+b+c+3)$ is a subspace of $\mathbb{R}^3$.

   (h) If $\dim(V) = n$, then any set of $n-1$ vectors in V must be linearly independent.

   (i) If $A \in \mathbb{R}^{m\times n}$, rank$(A) \leq \min(m, n)$.

   (j) If a set S of vectors in V contains the zero vector, then S is linearly dependent.

2. [15] What are the maximum and minimum possible ranks of a 5×3 matrix. Give one example of a matrix of each type. For each of your examples write down its nullity.

3. [20] Let $\mathbf{x}_1, \cdots, \mathbf{x}_n \in \mathbb{R}^n$ are linearly independent and let $A \in \mathbb{R}^{n\times n}$ such that $\text{Rank}(A) = n$. Prove that if

$$\mathbf{y}_i = A\mathbf{x}_i$$

for $i = 1, \cdots, n$, then $\mathbf{y}_1, \cdots, \mathbf{y}_n$ are linearly independent.

4. [15] Show that the union of the x-axis and the z-axis in $\mathbb{R}^3$, i.e. the set $S = \{(x, y, z)^T \in \mathbb{R}^3 | x = 0 \text{ or } z = 0\}$, is not a subspace of $\mathbb{R}^3$.

5. [15] Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 4 & 4 & 1 & 12 & 3 \\ -6 & -6 & -2 & -20 & -5 \\ 2 & 2 & 1 & 8 & 4 \\ -6 & -6 & -2 & -20 & -6 \end{pmatrix}.$$

By row operations A can be transformed into matrix B given by

$$B = \begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) What is the rank of A?

(b) Find a basis for the column space of A.

(c) Find a basis for the null space of A.

6. [15] Consider the vector space $\mathbb{R}^{2\times 2}$ and the following sets of ordered basis

$$\begin{aligned} E &= \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\} \\ F &= \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \end{aligned}$$

(a) Write down the coordinate vector for each element in E with respect to the basis F.

(b) Find the transition matrix U from the basis E to the basis F.

End of examination
Total pages: 5
Total marks: 100