Math 313 exam for honors linear algebra.

1. Suppose V is a vector space over the field F and $v_1, v_2, \ldots, v_r$ are nonzero eigenvectors associated to distinct eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_r$ of $T \in L(V,V)$. Prove the vectors $v_1, v_2, \ldots, v_r$ are linearly independent.

2. Let $V = \mathbb{R}^2$ be the plane and let $\mathcal{F} = \mathcal{F}(V)$ be the vector space of (continuous) functions. A vector $f \in \mathcal{F}$ is a function $f : V \longrightarrow \mathbb{R}$.

(i) If $T \in L(V,V)$, prove the transformation M_T of $\mathcal{F}$ which transforms the function $f \in \mathcal{F}$ to the function $(M_T(f))(x) = f(T(x))$ is a linear transformation. In other words show $M_T \in L(\mathcal{F}, \mathcal{F})$.

T **F** The condition that $T \in L(V,V)$ is not really necessary. The result is also true if $T : V \longrightarrow V$ is a (continuous) function.

(ii) Prove $M_T \circ M_S = M_{ST}$. Note the reversed order. Determine when M_T is invertible.

(iii) Suppose R_L is the reflection across a 1-dimensional subspace L of V. Describe in terms of their graphs, the vectors $f \in \mathcal{F}$ with the property $M_{R_L}(f) = f$. Similarly, describe $f \in \mathcal{F}$ with the property $M_{R_L}(f) = -f$.

3. Let $V = \mathbb{R}^n$, with inner/dot product $\langle\, , \rangle$. If W is a subspace of V, recall the definition

$$W^\perp := \{ v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W \} .$$

(i) Recall $T \in L(V,V)$ is self adjoint if $\langle T(v), w \rangle = \langle v, T(w) \rangle$ for every $v, w \in V$. If T is self adjoint, and $v_1, v_2, \ldots, v_n$ is an orthonormal basis of V, prove the matrix of T in this basis is symmetric.

(ii) If $T \in L(V,V)$ is self adjoint, and W is a T-invariant subspace of V, prove the subspace $W^\perp$ is also T-invariant.

(iii) If $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ is a self adjoint transformation of the plane, prove it is possible to find an orthonormal eigenbasis w_1, w_2 of $\mathbb{R}^2$.

[Hint. Regardless of whether you answered part (i), you may assume that, in the usual orthonormal basis, the matrix of T is symmetric. Explain why: (1) the roots of characteristic polynomial are real, (2) there is an eigenbasis, and (3) if w_1, w_2 are eigenvectors of T with distinct eigenvalues λ_1, λ_2, then justify $\lambda_1 \langle w_1, w_2 \rangle = \langle T(w_1), w_2 \rangle = \langle w_1, T(w_2) \rangle = \lambda_2 \langle w_1, w_2 \rangle$ and use it to say $w_1 \perp w_2$.]

4. Suppose F is a field, and V is a finite dimensional vector space over F.

(i) If $T \in L(V,V)$, explain the meaning of the term minimal polynomial of T. If $m(x)$ is a minimal polynomial of T, and $f(x)$ is a polynomial with $f(T) = 0$, prove $m(x)$ divides $f(x)$.

(ii) If $m(x)$ is a minimal polynomial of T and $m(x)$ factors as $m(x) = f(x)g(x)$ with $\gcd(f(x), g(x)) = 1$, so there are polynomials $a(x)$ and $b(x)$ satisfying $1 =$

$a(x)f(x) + b(x)g(x)$, prove the linear transformations $E_1 = a(T)f(T)$, $E_2 = b(T)g(T)$ have the property that $I = E_1 + E_2$, $E_1E_2 = 0 = E_2E_1$.

5. Calculations I

(i) Express the greatest common divisor of 39 and 65 as an integer linear combination of 39 and 65.

(ii) Let $\mathbb{F}_2 = \{0, 1\}$ be the binary field. Explain how you would construct a field with $8 = 2^3$ elements using a polynomial in $\mathbb{F}_2[x]$. **Find such a polynomial.** Do **not** spend your time writing out the addition and multiplication tables. Instead compute an example as follows: (1) Write out the 8 elements, (2) pick two elements a, b of degree 2, and determine which of your 8 elements equals the product ab.

(iii) Consider the three vectors

$$f_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad , \quad f_2 = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad , \quad v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} .$$

Find the (usual x, y, z coordinates of the) vector $w \in \mathrm{Span}(f_1, f_2)$ which is as close as possible to the vector v.

6. Calculations II

(i) Take $\mathcal{F}$ to be subspace of $\mathbb{R}[x]$ spanned by the polynomials $1, x, x^2$. The function $T : \mathcal{F} \longrightarrow \mathcal{F}$ defined as $T(f(x)) := x^2 f''(x)$ (the notation $f''(x)$ means the 2nd derivative) is a linear transformation. Determine the minimal polynomial of T.

(ii) Given the information:

$$A = \begin{bmatrix} 3 & 7 & -5 \\ -2 & -5 & 4 \\ -1 & -3 & 3 \end{bmatrix}$$

$$\mathrm{rref}\,(A) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathrm{rref}\,(A^k) = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{for } k \geq 2$$

$$\mathrm{rref}\,((A - I)^k) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{for } k \geq 1$$

Determine a basis which exhibits the Jordan canonical form of A and find the Jordan canonical form. Determine the minimal and characteristic polynomials of A.