Math 511 HW 1. Due Wednesday September 25, 2002

This is the final version of this homework set. It has 7 problems.

1. Suppose m and n are two positive integers.
   (i) If m and n are relatively prime, show the direct product $C_m \times C_n$ of the cyclic groups C_m and C_n is isomorphic to the cyclic group C_{mn} of order mn.
   (ii) If m and n are NOT relatively prime, show the direct procduct $C_m \times C_n$ is NOT a cyclic group.

2. Consider the group GL(2) of invertible matrices with respect to some field F
   (i) Argue by row operations that any element of GL(2) is expressible either as

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix},$$

   and determine if there is any element which is expressible in both ways.
   (ii) Given an arbitrary nonzero b, show it is possible to select x, y, and z satisfying

$$\begin{bmatrix} 0 & \frac{-1}{b} \\ b & 0 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

   Conclude that if N is a normal subgroup of GL(2), which contains an element of the type

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \quad \text{with} \quad x \neq 0$$

   then N must contain the subgroup SL(2).

2old. THIS PROBLEM IS POSTPONED TO HW02. Prove or provide a counterexample to the statement that if N is a normal subgroup of G and H a normal subgroup of N, then H is a normal subgroup of G.

3. Let p be a prime integer.
   (i) Use the fact that $\mathbb{F}_p^\times$ is a group to prove that $a^{p-1} \equiv 1 \bmod p$ for every integer a not congruent to zero.
   (ii) Prove Fermat's Theorem: For every integer a,

$$a^p \equiv a \bmod p$$

   (iii) Prove Wilson's Theorem:

$$(p-1)! \equiv -1 \bmod p$$

4. If $A, B \in \mathrm{GL}(n, \mathbb{R})$, define $A \sim B$ provided AB^{-1} is an orthogonal matrix.

T F $\sim$ is an equivalence relation.

Justify your answer.

5. Let

$$\mathrm{SL}(2,\mathbb{Z}) \;=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=1 \;\right\}$$

(i) Prove the homomorphism $\mathrm{SL}(2,\mathbb{Z}) \longrightarrow \mathrm{SL}(2,\mathbb{F}_p)$ given by reduction modulo p is surjective.

(ii) Determine all possible characteristic polynomials of elements $g \in \mathrm{SL}(2,\mathbb{Z})$ which have finite order. [Hint: The two conditions, $g \in \mathrm{SL}(2,\mathbb{Z})$ and g of finite order place constraints of the coefficients of the characteristic ploynomial.]

(iii) T F It is possible to find an element $g \in \mathrm{SL}(2,\mathbb{Z})$ of order 5.

6. Suppose V is a finite dimensional vector space over a field F, $T : V \longrightarrow V$ is a linear transformation, and $m(x) = p_1(x)^{e_1} \cdots p_r(x)^{e_r}$ is a factorization of the minimal polynomial of T into distinct irreducible polynomial factors. Set

$$q_i(x) = \frac{m(x)}{p_i(x)^{e_i}} \;.$$

Then $\gcd(\, q_1(x)\,,\, \ldots\,, q_r(x)\,) \;=\; 1$, so there exists polynomials $a_i(x)$ such that

$$1 \;=\; a_1(x)q_1(x) + \cdots + a_r(x)q_r(x) \;.$$

Set $E_i = a_i(T)q_i(T)$. Notice that $I = E_1 + \cdots + E_r$. Prove:

(i) $E_iE_j = 0$ if $I \neq j$ and $E_i = {E_i}^2$.

(ii) The image of E_i, i.e. $E_i(V)$ equals the kernel of $p_i(T)^{e_i}$.

7. In GL(n), set U to be the subgroup of upper triangular matrices with diagonal entries all 1. Suppose $M \in \mathrm{GL(n)}$ is a permutation matrix. Prove $M\,U\,M^{-1} = U$ if and only if M is the identity permutation.