Math 511 HW 2. Due Monday October 21, 2002

This is the final version of this homework set. It has 7 problems.

1. Prove or provide a counterexample to the statement that if N is a normal subgroup of G and H a normal subgroup of N, then H is a normal subgroup of G.

2. Prove that the ring $M_n(\mathbb{C})$ of n×n matrices is a simple r ing. That is the only ideals in $M_n(\mathbb{C})$ are either $\{0\}$ or $M_n(\mathbb{C})$.

3. If $I_1, \ldots, I_n$ are ideals in a commutative ring R, the product $I_1 \cdots I_n$ is defined as the set of all finite sums $\sum_i a_{i,1}a_{i,2}\ldots a_{i,n}$, where $a_{i,j} \in I_j$.
   (i) Prove the product set is an ideal.
   (ii) If R satisfies the hypothesis of the Chinese remainder theorem prove
   $$I_1 \cap \ldots \cap I_n = I_1 \cdots I_n$$

4. In the polynomial ring $\mathbb{Q}[x, y]$, determine if the ideal $I := (x^2 + 1, y^2 + 1)$ generated by $x^2 + 1$ and $y^2 + 1$ is a maximal ideal. Prove your answer.

5. Let $R = \mathbb{Z}[\sqrt{-5}]$ denote the ring of all complex numbers of the form $a + b\sqrt{-5}$ with $a, b \in \mathbb{Z}$. Find an element $r \in R$ which can be written in two distinct ways as a product of (nonassociate) irreducible elements and prove your factors are irreducible.

6. Let $R = \mathbb{Z}[\sqrt{-5}]$ denote the ring of all complex numbers of the form $a+b\sqrt{-5}$ with $a, b \in \mathbb{Z}$. Consider the two principal ideals $I_1 = 2R$, and $I_2 = (1 + \sqrt{-5})R$. Set $I := I_1 + I_2$.
   (i) Prove the ideal I is not a principal ideal.
   (ii) Compute R/I; in particular, prove I is a maximal ideal.
   (iii) Prove the product ideal $I \cdot I$ is a principal ideal cR and find c.

7. Suppose R is a commutative ring. The ring $R[[x]]$ of formal power series is the ring with elements
   $$\sum_{k=0}^{\infty} a_k\, x^k \qquad a_k \in R$$
   and addition, multiplication defined by
   $$\sum_{k=0}^{\infty} a_k\, x^k + \sum_{k=0}^{\infty} b_k\, x^k = \sum_{k=0}^{\infty} (a_k + b_k)\, x^k$$
   $$\Big(\sum_{k=0}^{\infty} a_k\, x^k\Big) \cdot \Big(\sum_{k=0}^{\infty} b_k\, x^k\Big) = \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} (a_j b_{k-j})\Big)\, x^k$$
   If $R = K$ is a field, determine all the ideals of $K[[x]]$. [Hint: If the power series $f = \sum_{k=0}^{\infty} a_k\, x^k$ has $a_0 \neq 0$, then f has an inverse in $K[[x]]$.]