Math 511 HW 3. Due Monday November 18, 2002

This is the final version of this homework set. It has 7 problems.

1. Consider the ring $R := \mathbb{Z}/6\mathbb{Z}$. The ring is not an integral domain since $(2+6\mathbb{Z})\cdot(3+6\mathbb{Z}) = (0+6\mathbb{Z})$. The set $S := \{\, 1+6\mathbb{Z}\,,\ 2+6\mathbb{Z}\,,\ 4+6\mathbb{Z}\,\}$ is a multiplicative subset of R. Determine the ring $S^{-1}R$ obtained by applying the localization procedure to $R \times S$. Note: By definition, $(a,b) \sim (c,d)$ means there is $s \in S$ such that $s(ad - bc) = 0$.

2. Determine how many 3×3 nilpotent matrices there are in $\mathrm{M}_3(\mathbb{F}_q)$. A matrix is nilpotent if its characteristic polynomial $p(t)$ is a power of t.

3. If the group G acts as automorphisms on the set X, we say the action is doubly transitive if given $x_1 \neq x_2$ and $y_1 \neq y_2$ there is an element $g \in G$ so that $g \cdot x_1 = y_1$ and $g \cdot x_2 = y_2$. Let G be a group that acts doubly transitively on a finite set X and fix $x \in X$.
   (i) Prove that the stabilizer subgroup $\mathrm{Stab}(x) = \{g \in G \mid g \cdot x = x\}$ is a maximal subgroup of G.
   (ii) Consider the special case of the action of the group of invertible linear transformations $\mathrm{GL}(2,\mathbb{R})$ on one-dimensional subspaces of $\mathbb{R}^2$. Prove this action is doubly transitive. By (i), the stabilizer of a line is a maximal subgroup of G. Compute the stabilizer of the span of the usual (column) basis vector e_1.

4. There are $70 = \binom{8}{4}$ ways to color the edges of a regular octagon, coloring four edges red and four edges yellow. The group D_8 of rotation and reflectional symmetries of the octagon has order 16 and acts on the 70 colorings. Determine the number and sizes of the orbits.

5. Determine, up to isomorphism, all the nonabelian subgroups of order 12.

6. The group $\mathrm{GL}(2,\mathbb{F}_3)$ of 2×2 invertible matrices has order $(3^2-1)(3^2-3) = 48$. Determine a composition series for this group. [Hint. How many 1-dimensional subspaces are there in $\mathbb{F}_3^2$? This yields a homomorphism of $\mathrm{GL}(2,\mathbb{F}_3)$ to the permutation group of the set of 1-dimenional subspaces and therefore a reduction in the problem.]

7. Suppose A is a finite abelian group whose order $|A|$ has prime factorization $p_1^{r_1}\cdots p_s^{r_s}$. Denote by P_i, the p_i-Sylow subgroup of order $p_i^{r_i}$. Prove every element of $a \in A$ is expressible uniquely as $a = a_1 + \cdots + a_s$ $(a_i \in P_i)$. This says A is the direct sum of its Sylow subgroups.