Math 511 HW 4. Due Friday December 13, 2002

This is the final version of this homework set. It has 8 problems.

1. Determine up to isomorphism all the abelian groups of order 675.

2. Find a direct sum of cyclic groups which is isomorphic to the free abelian group generated by four generators u, v, w, x subject to the relations

$$\begin{array}{ccccccccccc} u & + & v & + & w & - & 2x & = & 0 \\ u & + & 2v & + & w & + & 2x & = & 0 \\ 2u & + & 3v & + & 2w & + & 0 & = & 0 \\ 2u & + & 0 & + & w & + & 2x & = & 0 \end{array}$$

3. Given a ring R, suppose M is a right R-module and N is a left R-module. An example of this is the ring of square matrices $R = M_{a\times a}(\mathbb{C})$, and M (resp. N)) is the right (resp. left) R-module of row (resp. column) vectors of size a. The tensor product (over R) of M and N is the abelian group defined as follows.
   - Let F denote the free $\mathbb{Z}$-module which has a basis $1_{(m,n)}$ indexed by pairs in $M \times N$.
   - Let G be the $\mathbb{Z}$-module generated by (all) the elements
     (1) $1_{(m_1+m_2,n)} - 1_{(m_1,n)} - 1_{(m_2,n)}$, as well as $1_{(m,n_1+n_2)} - 1_{(m,n_1)} - 1_{(m,n_2)}$
     (2) $1_{(mr,n)} - 1_{(m,rn)}$

     Here, $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$.
   - The **tensor product** is defined to be the quotient $\mathbb{Z}$-module F/G.

   (i) Suppose B is a $\mathbb{Z}$-module and $f : M \times N \longrightarrow B$ with the properties (i1) $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$, (i2) $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$, and (ii) $f(mr, n) = f(m, rn)$. Prove there is a unique $\mathbb{Z}$-module map $\theta : F/G \longrightarrow B$ so that $\theta(1_{(m,n)} + G) = f(m, n)$. Remember, the free $\mathbb{Z}$-module F has a universal property.

   (ii) View the cyclic groups $\mathbb{Z}/(3)$ and $\mathbb{Z}/(6)$ as right/left $\mathbb{Z}$-modules. Compute their tensor product.

   (iii) **T** **F** If we think of R^+, the additive group of R as a left R module, then for any right R-module M, the abelian groups M and the tensor product M with R^+ are isomomorphic. Justify your answer.

   (iv) Prove $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{Q}$.

4.

(i) If R is a ring, prove $M_{m\times m}(R)^{\text{opp}}$, is isomorphic to $M_{m\times m}(R^{\text{opp}})$.

(ii) Consider the two subrings of $M_{5\times 5}(\mathbb{C})$ defined as

$$R_1 := \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} \qquad R_2 := \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}$$

Verify that

$$I_1 := \begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad I_2 := \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

are ideals in R_1 and R_2 respectively.

T **F** R_1 and R_2 are isomorphic. Justify your answer.

T **F** R_1^{opp} and R_2 are isomorphic. Justify your answer.

Hint: It is known that if $\theta : R_1 \longrightarrow R_2$ is an isomorphism (or anti-isomorphism), then θ must take I_1 onto I_2. You may assume this. Properties of the transpose, and the use of the anti-identity matrix are helpful.

(iii) **Extra credit** Prove the hint.

5. In this problem you will show the group $\mathrm{SL}(2, \mathbb{F}_5)/\{\pm I\}$ is isomorphic to the group A5. Recall the order of $\mathrm{GL}(2, \mathbb{F}_p)$ is $(p^2 - 1)(p^2 - p)$ and that of $\mathrm{SL}(2, \mathbb{F}_p) = \ker(\det)$ is $(p^2 - 1)p$, so the order of $\mathrm{SL}(2, \mathbb{F}_5)$ is $120 = 2^3 \cdot 3 \cdot 5$.

(i) By the Sylow Theorem, $\mathrm{SL}(2, \mathbb{F}_5)$ contains a subgroup of order 2^3. Find one. Remember all ℓ-Sylow subgroup are conjugate. [Hint. What is the order of the subgroup of diagonal matrices?]

(ii) Determine the structure of your answer in part (i). In particular, list the number of elements of order 2 and 4.

(iii) Use properties of characteristic polynomials and your answer to part (ii) to prove $\mathrm{SL}(2, \mathbb{F}_5)$ contain a subgroup H of size $2^3 \cdot 3$. Hence, H has index 5 in $\mathrm{SL}(2, \mathbb{F}_5)$. Use this to exhibit an homomorphism $\theta : \mathrm{SL}(2, \mathbb{F}_5) \longrightarrow$ S5. Prove the image has size 60, and conclude $\mathrm{SL}(2, \mathbb{F}_5)/\{\pm I\}$ is isomorphic to the group A5.

6. Suppose R is a possibly non-commutative ring. An R-module is called simple if it is not the zero module and it has no proper R-submodule.
   (i) Prove that any simple module is isomorphic to R/I where I is a maximal left ideal.
   (ii) Prove Schur's Lemma: Let $\phi : S \longrightarrow S'$ be a homorphism of simple modules. Then either ϕ is zero, or else it is an isomorphism.

7. (Artin §12.8 #2 page 489) Let $R = \mathbb{C}[x, y]$, and let M be the finitely generated R-module which is generated by elements a, b, c and d subject to the two relations

$$\begin{array}{ccccccccc} a & + & yb & + & xc & + & x^2d & = & 0 \\ xa & + & (x+3)b & + & yc & + & y^2d & = & 0 \end{array}$$

   This means M is the quotient of a free R-module $\tilde{M}$ with basis $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, by the R-submodule $\tilde{K}$ generated by the two elements

$$\tilde{a} + y\tilde{b} + x\tilde{c} + x^2\tilde{d} \qquad \text{and} \qquad x\tilde{a} + (x+3)\tilde{b} + y\tilde{c} + y^2\tilde{d}\ .$$

   (i) Prove M is a free R-module.
   (ii) Prove that for any $\alpha\,,\,\beta \in \mathbb{C}$, the matrix

$$\begin{bmatrix} 1 & \beta & \alpha & \alpha^2 \\ \alpha & (\alpha+3) & \beta & \beta^2 \end{bmatrix}$$

   has rank 2. [Hint. Your solution part (i) should yield an easy solution to part (ii)]. It is rather amazing that there is a general but difficult result that says the truth of (ii) implies that of (i). See Artin §12.8.]

8. Let $R = \mathbb{Z}[\sqrt{-5}]$. Remember that R is not a unique factorization domain. $2 \cdot 3 = (1+\sqrt{-5}) \cdot (1-\sqrt{-5})$.
   (i) Prove the ideal $I := 2R + (1+\sqrt{-5})R$ is not a free module.
   (ii) Let R^2 be the free R-module of size 2 row vectors over R, and let $\theta : R^2 \longrightarrow I$ be the R-homomorphism

$$\theta[\,r_1\,,\,r_2\,] \;=\; 2\,r_1 \;+\; (1+\sqrt{-5})r_2\ .$$

   Prove the R-submodule $\ker(\theta)$ is not a free R-submodule of R^2. [Hint. Note that both $[3, -(1+\sqrt{-5})]$ and $[(1+\sqrt{-5}), -2]$ both belong to the kernel.]
   (iii) Prove there is an R-monomorphism $\sigma : I \longrightarrow R^2$ so that the composite map $\theta \circ \sigma$ is the identity map of I. Since σ is a monomorphism, I and $\sigma(I)$ are isomorphic R-modules. In your answer specify $\sigma(2)$ and $\sigma(1+\sqrt{-5})$. Conclude that the free R-module R^2 of rank 2 is the direct sum of the two submodules $\ker(\theta)$ and $\sigma(I)$, each of which is NOT a free R-module of rank 1. Recall that a submodule of a free finite rank d module over a PID is free of rank at most d. This unusual but interesting phenomenon exhibited by the non-PID $\mathbb{Z}[\sqrt{-5}]$, leads to interesting studies. The two modules $\ker(\theta)$ and $\sigma(I) \cong I$ are called projective modules. Projective modules are generalizations of free modules.