Duration and Portfolio Immunization

Macaulay duration

The duration of a fixed income instrument is a weighted average of the times that payments (cash flows) are made. The weighting coefficients are the present values of the individual cash flows.

$$D = \frac{PV(t_0)t_0 + PV(t_1)t_1 + \cdots + PV(t_n)t_n}{PV}$$

where $PV(t)$ denotes the present value of the cash flow that occurs at time t.

If the present value calculations are based on the bond's yield, then it is called the *Macaulay duration*.

Let P denote the price of a bond with m coupon payments per year; also, let

y : yield per each coupon payment period,

n : number of coupon payment periods

F: par value paid at maturity

$\tilde{C}$: coupon amount in each coupon payment

Now, $$P = \frac{\tilde{C}}{1+y} + \frac{\tilde{C}}{(1+y)^2} + \cdots + \frac{\tilde{C}}{(1+y)^n} + \frac{F}{(1+y)^n}$$

then $$\frac{1}{P}\frac{dP}{d\lambda} = -\frac{1}{1+y}\left(\frac{1}{m}\right)\left[\frac{1\cdot\tilde{C}}{1+y} + \frac{2\cdot\tilde{C}}{(1+y)^2} + \cdots + \frac{n\tilde{C}}{(1+y)^n} + \frac{nF}{(1+y)^n}\right]\frac{1}{P}$$

Note that $\lambda = my$.

modified duration = $\dfrac{1}{P}\dfrac{dP}{d\lambda}$

Macaulay duration = $\dfrac{1}{m}\dfrac{\dfrac{nF}{(1+y)^n}+\sum_{k=1}^{n}\dfrac{k\tilde{C}}{(1+y)^k}}{P}$

- The negativity of $\dfrac{1}{P}\dfrac{dP}{d\lambda}$ indicates that bond price drops as yield increases.

- Prices of bonds with longer maturities drop more steeply with increase of yield.

  This is because bonds of longer maturity have longer Macaulay duration:

$$\frac{\Delta P}{P} \approx -\frac{D_{Mac}}{1+y}\Delta\lambda.$$

Example

Consider a 7% bond with 3 years to maturity. Assume that the bond is selling at 8% yield.

A	B	C	D	E	
Year	Payment	Discount factor 8%	Present value $= B \times C$	Weight = D/Price	$A \times E$
0.5	3.5	0.962	3.365	0.035	0.017
1.0	3.5	0.925	3.236	0.033	0.033
1.5	3.5	0.889	3.111	0.032	0.048
2.0	3.5	0.855	2.992	0.031	0.061
2.5	3.5	0.822	2.877	0.030	0.074
3.0	103.5	0.79	81.798	0.840	2.520
Sum		Price = 97.379		Duration = 2.753	

Here, $\lambda = 0.08$, $m = 2$, $y = 0.04$, n = 6, $\tilde{C} = 3.5$, $F = 100$.

Quatitative properties of duration

Duration of bonds with 5% yield as a function of maturity and coupon rate.

Coupon rate

Years to maturity	1%	2%	5%	10%
1	0.997	0.995	0.988	0.977
2	1.984	1.969	1.928	1.868
5	4.875	4.763	4.485	4.156
10	9.416	8.950	7.989	7.107
25	20.164	17.715	14.536	12.754
50	26.666	22.284	18.765	17.384
100	22.572	21.200	20.363	20.067
Infinity	20.500	20.500	20.500	20.500

Suppose the yield changes to 8.2%, what is the corresponding change in bond price?

Here, $y = 0.04$, $\Delta\lambda = 0.2\%$, $P = 97.379$, $D = 2.753$, $m = 2$.

The change in bond price is approximated by

$$\frac{1}{P}\frac{\Delta P}{\Delta \lambda} \approx -\frac{1}{1+y} \times D$$

i.e. $$\Delta P \approx -\frac{97.379 \times 0.2\% \times 2.753}{1.04}$$

Properties of duration

1. Duration of a coupon paying bond is always less than its maturity. Duration decreases with the increase of coupon rate. Duration equals bond maturity for non-coupon paying bond.

2. As the time to maturity increases to infinity, the duration do not increase to infinity but tend to a finite limit independent of the coupon rate.

   Actually, $D \to \dfrac{1+\frac{\lambda}{m}}{\lambda}$ where λ is the yield per annum, and m is the number of coupon payments per year.

- Durations are not quite sensitive to increase in coupon rate (for bonds with fixed yield).

- When the coupon rate is lower than the yield, the duration first increases with maturity to some maximum value then decreases to the asymptotic limit value.

Duration of a portfolio

Suppose there are m fixed income securities with prices and durations of P_i and D_i, $i = 1,2,\ldots, m$, all computed at a *common yield*. The portfolio value and portfolio duration are then given by

$$P = P_1 + P_2 + \ldots + Pm$$

$$D = W_1D_1 + W_2D_2 + \ldots + W_mD_m$$

where $W_i = \dfrac{P_i}{P_1 + P_2 + \cdots + P_m}, \qquad i = 1,2,\cdots,m.$

Example

Bond	*Market value*	*Portfolio weight*	*Duration*
A	\$10 million	0.10	4
B	\$40 million	0.40	7
C	\$30 million	0.30	6
D	\$20 million	0.20	2

$$\text{Portfolio duration} = 0.1 \times 4 + 0.4 \times 7 + 0.3 \times 6 + 0.2 \times 2$$
$$= 5.4.$$

Roughly speaking, if all the yields affecting the four bonds change by 100 basis points, the portfolio value will change by approximately 5.4%.

Management of bond portfolios

Suppose a corporation faces a series of cash obligations in the future and would like to acquire a portfolio of bonds that it will use to pay these obligations.

Simple solution (may not be feasible in practice)

Purchase a set of zero-coupon bonds that have maturities and face values exactly matching the separate obligations.

Immunization

- If the yields do not change, one may acquire a bond portfolio having a value equal to the present value of the stream of obligations. One can sell part of the portfolio whenever a particular cash obligation is required.
- A better solution requires matching the duration as well as present values of the portfolio and the future cash obligations.
- This process is called immunization (protection against changes in yield). By matching duration, portfolio value and present value of cash obligations will respond identically (to first order approximation) to a change in yield.

Difficulties with immunization procedure

1. It is necessary to rebalance or re-immunize the portfolio from time to time since the duration depends on yield.

2. The immunization method assumes that all yields are equal (not quite realistic to have bonds with different maturities to have the same yield).

3. When the prevailing interest rate changes, it is unlikely that the yields on all bonds all change by the same amount.

Example

Suppose Company A has an obligation to pay \$1 million in 10 years. How to invest in bonds now so as to meet the future obligation?

- An obvious solution is the purchase of a simple zero-coupon bond with maturity 10 years.

Suppose only the following bonds are available for its choice.

	coupon rate	maturity	price	yield	duration
Bond 1	6%	30 yr	69.04	9%	11.44
Bond 2	11%	10 yr	113.01	9%	6.54
Bond 3	9%	20 yr	100.00	9%	9.61

- Present value of obligation at 9% yield is \$414,643.

- Since Bonds 2 and 3 have durations shorter than 10 years, it is not possible to attain a portfolio with duration 10 years using these two bonds.

Suppose we use Bond 1 and Bond 2 of amounts V_1 & V_2,

$$V_1 + V_2 = PV$$
$$P_1 V_1 + D_2 V_2 = 10 \times PV$$

giving $V_1 = \$292,788.64$, $V_2 = \$121,854.78$.

Yield

	9.0	8.0	10.0
Bond 1			
Price	69.04	77.38	62.14
Shares	4241	4241	4241
Value	292798.64	328168.58	263535.74
Bond 2			
Price	113.01	120.39	106.23
Shares	1078	1078	1078
Value	121824.78	129780.42	114515.94
Obligation value	414642.86	456386.95	376889.48
Surplus	-19.44	1562.05	1162.20

Observation

At different yields (8% and 10%), the value of the portfolio almost agrees with that of the obligation.

Convexity measure

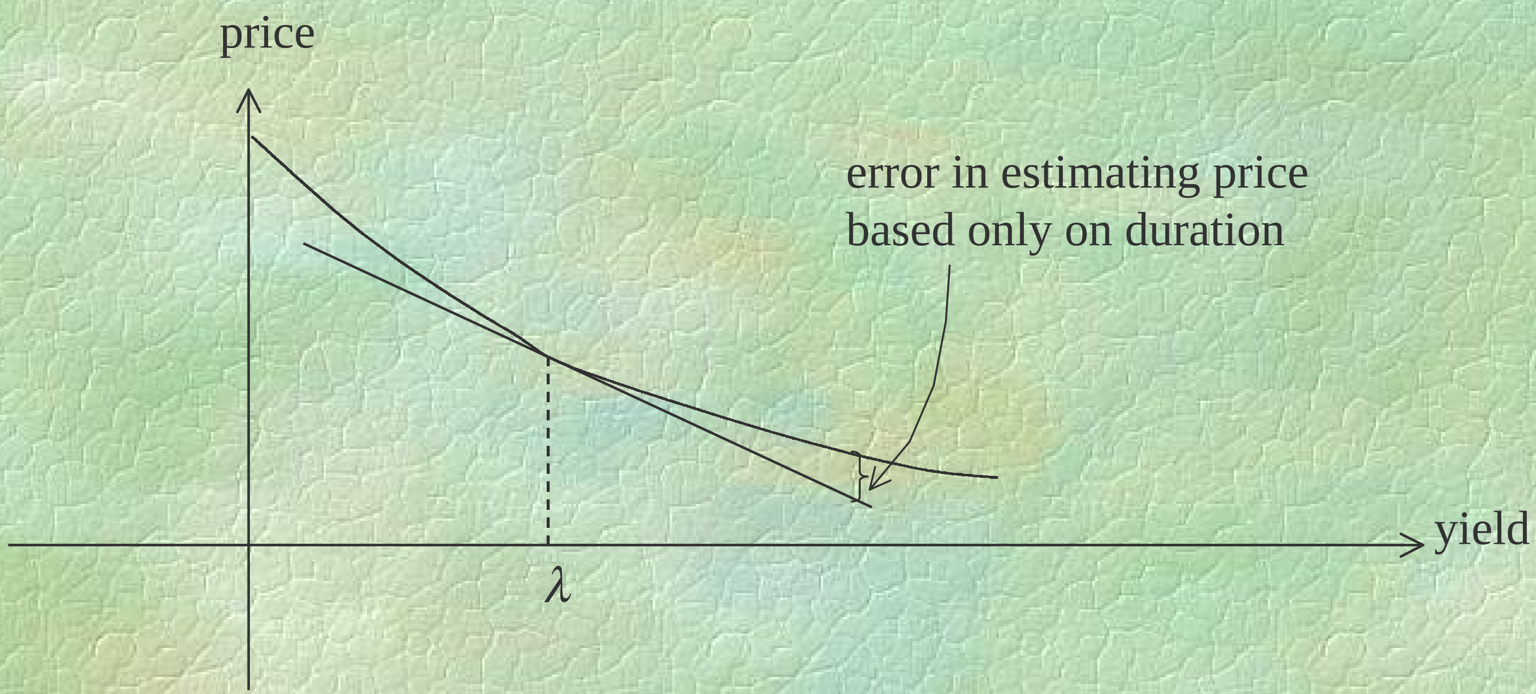
Taylor series expansion

$$\Delta P \approx \frac{dP}{d\lambda}\Delta\lambda + \frac{1}{2}\frac{d^2P}{d\lambda^2}(\Delta\lambda)^2 + \text{higher order terms.}$$

To first order approximation, the modified duration $\frac{1}{P}\frac{dP}{d\lambda}$ measures the percentage price change due to change in yield $\Delta\lambda$.

Zero convexity

This occurs only when the price yield curve is a straight line.



The convexity measure $\frac{1}{P}\frac{d^2P}{d\lambda^2}$ captures the percentage price change due to the convexity of the price yield curve.

Percentage change in bond price = $\frac{\Delta P}{P}$

$\approx$ modified duration $\times$ change in yield

+ convexity measure $\times$ (change in yield)^2/2