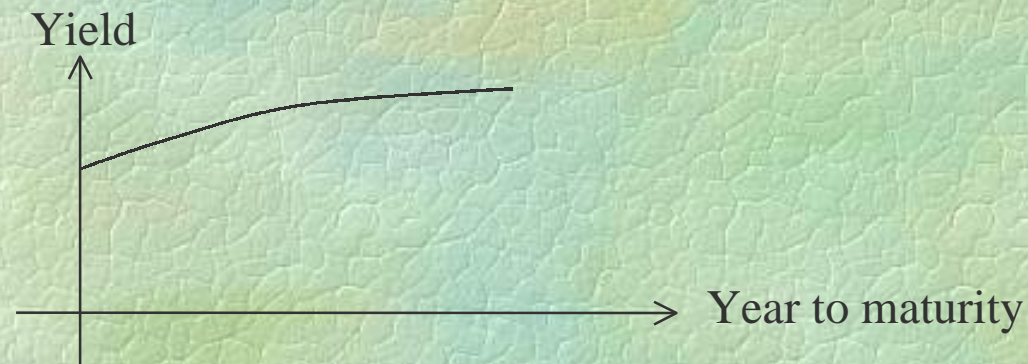
The background of the slide is a solid green color with a subtle, embossed texture that resembles leather or a fine, irregular pattern. The lighting appears slightly brighter in the center, creating a soft gradient.

Yield Curves and Term Structure Theory

Yield curve

The plot of yield on bonds of the same credit quality and liquidity against maturity is called a *yield curve*.



Remark The most typical shape of a yield curve has an upward slope.

The relationship between yields on otherwise comparable securities with different maturities is called the *term structure of interest rates*.

Ideally, yield curve should be plotted for bonds that are alike in all respects other than the maturity; but this is extremely difficult in practice. Bonds that have similar risks of default may be different in coupon rates, marketability, callability, etc.

Benchmark interest rate or base interest rate

Yield curve on US Treasury bond instruments is used to serve as a benchmark for pricing bonds and to set yields in other sectors of the debt market. This is because the US Treasury bonds are viewed as *default free* and they have the highest *liquidity*.

Yield spread and risk premium

On Sept 19, 1997, the yield on the Wal-Mart Stores bonds (rated AA) with 10 years to maturity was 6.476%. On the same date, the yield on the 10 year most recently issued Treasury was 6.086%.

$$\text{Yield spread} = 6.476\% - 6.086\% = 0.39\%.$$

This spread, called a *risk premium*, reflects the additional risks the investor faces by acquiring a security that is not issued by the US Government.

Term structure theory addresses how interest rates are charged depends on the length of time that the funds are held.

Spot rate

Spot rate is the yield on a zero-coupon Treasury security with the same maturity.

Any bond can be viewed as a package of zero-coupon instruments. It is not appropriate to use the same interest rate to discount all cash flows arising from the bond. Each cash flow should be discounted at a unique interest rate that is appropriate for the time period in which the cash flow will be received. That rate is the spot rate.

Example

A bank offers to depositors one-year spot rate of 4.5% and two-year spot rate of 5%.

That is, if you deposit \$100 today, you receive

- (i) \$104.5 in the one-year deposit one year later;
- (ii) $(1.05)^2 \times \$100 = \110.25 in the two-year deposit two years later.

Example

Given the spot rate curve for Treasury securities, find the fair price of a Treasury bond.

An 8% bond maturing in 10 years (par value = \$100).

Year	1	2	3	4	5	6	7	8	9	10	Total present value
spot rate (%)	5.571	6.088	6.555	6.978	7.361	7.707	8.020	8.304	8.561	8.793	
discount factor	0.947	0.889	0.827	0.764	0.701	0.641	0.583	0.528	0.477	0.431	
cash flow	8	8	8	8	8	8	8	8	8	108	
present value	7.58	7.11	6.61	6.11	5.61	5.12	4.66	4.23	3.82	46.50	97.34

Each cash flow is discounted by the discount factor for its time. For example, the discount factor of the coupon paid 8 years later is $\frac{1}{(1.08304)^8} \approx 0.528$.

Example (construction of a zero-coupon instrument)

Bond A: 10-year bond with 10% coupon; $P_A = \$98.72$.

Bond B: 10-year bond with 8% coupon; $P_B = \$85.89$.

Both bonds have the same par of \$100.

Construct a portfolio of -0.8 unit of bond A and 1 unit of bond B. Resulting face value is \$20, and price is $P_A - 0.8 P_B = 6.914$.

The coupon payments cancel, so this is a zero coupon portfolio. The 10-year spot rate is given by

$$(1 + S_{10})^{10} \times \$6.914 = \$20$$

giving $S_{10} = 11.2\%$.

Construction of spot rate curve

- The obvious way to determine a sport rate curve is to find the prices of a series of zero-coupon bonds with various maturity dates. However, “zero” with long maturities are rare.
- The spot rate curve can be determined from the prices of coupon-bearing bonds by beginning with short maturities and working forward toward longer maturities.

Example

Consider a two-year bond with coupon payments of amount C at the end of each year. The price is P_2 and the par value is F . Since the price should equal to the discounted value of the cash flow stream

$$P_2 = \frac{C}{1 + S_1} + \frac{C + F}{(1 + S_2)^2}$$

where S_1 and S_2 are the spot rates for one-year period and two-year period, respectively.

First, we determine S_1 by direct observation of one-year zero-coupon Treasury bill rate; then solve for S_2 algebraically from the above equation. The procedure is repeated with bonds of longer maturities, say,

$$P_3 = \frac{C}{1+S_1} + \frac{C}{(1+S_2)^2} + \frac{C+F}{(1+S_3)^3}.$$

- Note that Treasury bonds (considered to be default free) are used to construct the benchmark spot rates.

Forward rates

Forward rates are interest rates for money to be borrowed between two dates in the future but under terms agreed upon today.

Assume that the one-year and two-year spot rates, S_1 and S_2 , are known.

1. Buy a two-year bond

\$1 in a 2-year account will grow to $\$(1 + S_2)^2$ at the end of 2 years.

2. Buy a one-year bond and when it matures in one year from now, buy another one-year bond for another year.

Let f denote the *forward rate* between one year and two years agreed upon now. The investment will grow to $\$(1 + S_1)(1 + f)$ at the end of two years.

By *no arbitrage* principle, these two investments should have the same returns (if otherwise, one can long the higher return investment and sell short the lower return one).

Hence,

$$(1 + S_1)(1 + f) = (1 + S_2)^2$$

giving

$$f_{1,2} = \frac{(1 + S_2)^2}{1 + S_1} - 1.$$

This forward rate $f_{1,2}$ is implied by the two spot rates S_1 and S_2 .

Forward rate formulas

The implied forward rate between times t_1 and t_2 ($t_2 > t_1$) is the rate of interest between those times that is consistent with a given spot rate curve.

(1) *Yearly compounding*

$$(1 + S_j)^j = (1 + S_i)^i (1 + f_{i,j})^{j-i}, \quad j > i$$

giving

$$f_{i,j} = \left[\frac{(1 + S_j)^j}{(1 + S_i)^i} \right]^{1/(j-i)} - 1.$$

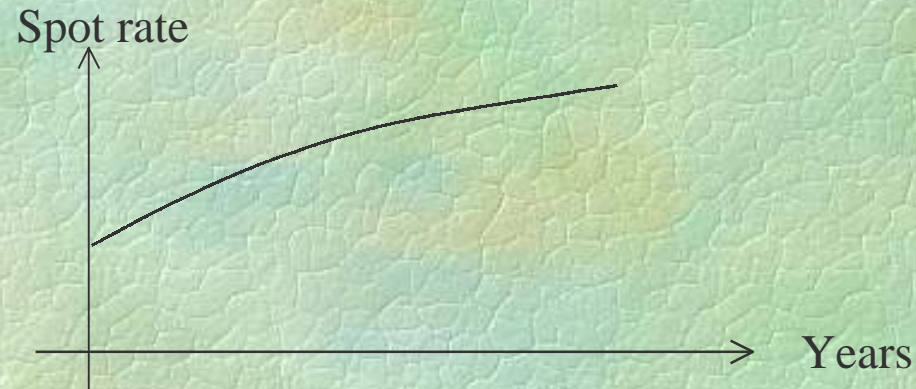
(2) *Continuous compounding*

$$e^{S_{t_2} t_2} = e^{S_{t_1} t_1} e^{f_{t_1, t_2} (t_2 - t_1)}$$

so that

$$f_{t_1 t_2} = \frac{S_{t_2} t_2 - S_{t_1} t_1}{t_2 - t_1}.$$

Determinants of term structure of interest rates



Most spot rate curves slope rapidly upward at short maturities and continue to slope upward but more gradually as maturities lengthen.

Three theories are proposed to explain the evolution of spot rate curves:

1. Expectations;
2. Liquidity preference;
3. Market Segmentation.

Expectations theory

From the spot rates S_1, \dots, S_n for the next n years, we can deduce a set of forward rates $f_{1,2}, \dots, f_{1,n}$. According to the expectations theory, these forward rates define the expected spot rate curves S_1', \dots, S_{n-1}' for the next year.

For example, suppose $S_1 = 7\%$, $S_2 = 8\%$, then $f_{1,2} = \frac{(1.08)^2}{1.07} - 1 = 9.01\%$.

Then this value of 9.01% is the market's expected value of next year's one-year spot rate S_1' .

Turn the view around: The expectation of next years curve determines what the current spot rate curve must be. That is, expectations about future rates are part of today's market.

Weakness

According to this hypothesis, then the market expects rates to increase whenever the spot rate curve slopes upward. Unfortunately, rates do not go up as often as expectations would imply.

Liquidity preference

- For bank deposits, depositors usually prefer short-term deposits over long-term deposits since they do not like to tie up capital (liquid rather than tied up). Hence, long-term deposits should demand high rates.
- For bonds, long-term bonds are more sensitive to interest rate changes. Hence, investors who anticipate to sell bonds shortly would prefer short-term bonds.

Market segmentation

The market for fixed income securities is segmented by maturity dates.

- To the extreme, all points on the spot rate curves are mutually independent. Each is determined by the forces of supply and demand.
- A modification to the extreme view is that adjacent rates cannot become grossly out of line with each other.

Expectations Dynamics

The expectations implied by the current spot rate curve will actually be fulfilled.

- To predict next year's spot rate curve from the current one under the above assumption. Given S_1, \dots, S_n as the current spot rates, how to estimate next year's spot rates $S'_1, S'_2, \dots, S'_{n-1}$. Recall that the current forward rate $f_{1,j}$ can be regarded as the expectation of what the interest rate will be next year, that is,

$$S'_{j-1} = f_{1,j} = \left[\frac{(1 + S_j)^j}{1 + S_1} \right]^{1/(j-1)} - 1.$$

Example

	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇
current	6.00	6.45	6.80	7.10	7.36	7.56	7.77
forecast	6.90	7.20	7.47	7.70	7.88	8.06	

$$f_{1,2} = \frac{(1.0645)^2}{1.06} - 1 = 0.069$$

$$f_{1,3} = \left[\frac{(1.068)^3}{1.06} \right]^{1/2} - 1 = 0.072$$

Invariance theorem

Suppose that interest rates evolve according to the expectation dynamics. Then a sum of money invested in the interest rate market for n years will grow by a factor $(1 + S_n)^n$, independent of the investment and reinvestment strategy (so long as all funds are fully invested).

This is not surprising since every investment earns the relevant short rates over the period of investment (short rates do not change under the expectations dynamics).

To understand the theorem, take $n = 2$.

1. Invest in a 2-year zero-coupon bond;
2. Invest in a 1-year bond, then reinvest the proceed at the end of the year.

The second strategy would lead as a growth of

$$(1 + S_1)(1 + f_{1,2}) = (1 + S_1) \left[\frac{(1 + S_2)^2}{1 + S_1} \right] = (1 + S_2)^2;$$

the same growth as that of the first strategy.

Discount factors between two times

Let $d_{j,k}$ denote the discount factor used to discount cash received at time k back to an equivalent amount of cash at time j ($j < k$).

We then have

$$d_{j,k} = \left[\frac{1}{1 + f_{j,k}} \right]^{k-j}$$

and these discount factors observe the compounding rule:

$$d_{i,k} = d_{i,j} \cdot d_{j,k}.$$

Short rates

Short rates are the forward rates spanning a single time period. The short rate at time k is $r_k = f_{k, \underline{k}+1}$.

The spot rate S_k and the short rates r_0, \dots, r_{k-1} are related by

$$(1 + S_k)^k = (1 + r_0) (1 + r_1) \dots (1 + r_{k-1})$$

In general,

$$(1 + f_{i,j})^{j-i} = (1 + r_i) (1 + r_{i+1}) \dots (1 + r_{j-1}).$$

The short rate for a specific year does not change (in the context of expectations dynamics).