Jarrow-Lando-Turnbull model

Characteristics

- Credit rating dynamics is represented by a Markov chain.
- Default is modelled as the first time a continuous time Markov chain with K states hitting the absorbing state K (default state).
- LGD is characterized as a fraction of an otherwise similar default-free claim.

Markov chain model

- To describe the dynamics of bond credit ratings

  Let X_t represent the credit rating at time t of a bond, and $X = \{X_t\ , t = 0, 1, 2, \ldots\}$ is a time-homogeneous Markov chain on the state space $N = \{1, 2, \ldots, K, K + 1\}$, $K + 1$ designates default (absorbing state)

$$Q = \begin{pmatrix} q_{11} & \cdots & q_{1k} & q_{1,K+1} \\ \vdots & & \vdots & \vdots \\ q_{k1} & & q_{KK} & q_{K,K+1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$q_{ij} = \Pr\left[X_{t+1} = j \middle| X_t = i\right], \quad i, j \in N, t = 0,1,2,\cdots$$

Assumptions on the risk premia

The credit rating process $\tilde{X}$ after risk neutralization is not necessarily Markovian

$$\tilde{q}_{ij}(t, t+1) = \pi_{ij}(t) q_{ij}$$

where q_{ij} is the actual transitional probabilities of the observed time-homogeneous Markov chain X,

$\pi_{ij}(t)$ are the risk premium adjustments.

JLT model assumes $\pi_{ij}(t) = \pi_i(t)$ for $j \neq i$, and they are deterministic functions of t. Some structure is imposed to improve analytical tractability.

The assumption is imposed to facilitate statistical information since the historical q_{ij} can be used in the inference process.

Risk neutralized transition matrix

Risk neutralized process $\tilde{X}$ becomes a non-homogeneous Markov chain with transitional probabilities

$$\tilde{q}_{ij}(t,t+1)=\begin{cases}\pi_i(t)q_{ij} & i\neq j\\ 1-\pi_i(t)(1-q_{ij}) & i=j\end{cases}$$

- $\tilde{X}$ and $\{r(t)\}$ (spot rate process) are assumed to be mutually independent under the risk neutral measure (accuracy may deteriorate for speculative-grade bonds).

Price of risky discount bonds

Risky discount bond in the j^{th} credit rating class

$$v_j(t,T) = \tilde{E}_t\left\{ e^{-\int_t^T r(s)ds}\left[\mathbf{1}_{\{\tau_j > T\}} + \delta \mathbf{1}_{\{\tau_j \le T\}} \right]\right\}$$

$$= \tilde{E}_t\left[e^{-\int_t^T r(s)ds}\right] \tilde{E}_t\left[\mathbf{1}_{\{\tau_j > T\}} + \delta \mathbf{1}_{\{\tau_j \le T\}} \right] \text{ (independence)}$$

$$= v_0(t,T)\left[\delta + (1-\delta)\tilde{P}_t\{\tau_j > T\}\right]$$

where τ_j is the absorption (default time) of $\tilde{X}$ when $\tilde{X}_t = j$.

$$\tilde{P}_t\left\{\tau_j > T\right\} = \sum_{k=1}^{K} \tilde{q}_{j,k}(t,T) = 1 - \tilde{q}_{j,K+1}(t,T).$$

Numerical implementation of Jarrow-Lamdo-Turnbull model

- modelling default and credit migration in preference to modelling recovery rate

$$d = \begin{matrix} I \\ J \\ D \end{matrix} \begin{pmatrix} 0.90 & 0.05 & 0.05 \\ 0.10 & 0.80 & 0.10 \\ 0 & 0 & 1.00 \end{pmatrix}$$

I = investment grade, J = junk grade and D = default (absorbing)

$$\begin{pmatrix} r_{01} \\ r_{02} \end{pmatrix} = \begin{pmatrix} 0.08 \\ 0.09 \end{pmatrix}, \begin{pmatrix} s_{I,01} \\ s_{I,02} \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.015 \end{pmatrix}, \begin{pmatrix} s_{J,01} \\ s_{J,02} \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0.03 \end{pmatrix}$$

- assume that there is no correlation between rating migration and interest rate

Pricing of risky debt of maturity one and two periods are

$$B_I(0,1) = \frac{1}{1.09}, \quad B_I(0,2) = \frac{1}{1.105^2}, \quad B_J(0,1) = \frac{1}{1.10}, \quad B_J(0,2) = \frac{1}{1.12^2}.$$

Assume recovery rate $\phi = 0.40$ so that payoff vector $= \begin{pmatrix} 1 \\ 1 \\ \phi \end{pmatrix}$.

Suppose currently at state I,

$$d_I = \begin{pmatrix} 0.90 \\ 0.05 \\ 0.05 \end{pmatrix}.$$

Transform d_I into the risk neutral vector q_I with adjustment π_I

$$q_I = \begin{pmatrix} 1 - 0.10\pi_I \\ 0.05\pi_I \\ 0.05\pi_I \end{pmatrix}.$$

We find π_I by making the expected value of discounted cash flow equal to the traded price of bond

$$B_I(1,2) = \frac{1}{1+r_{01}} C^T q_I$$

$$\frac{1}{1.09} = \frac{1}{1.08}(1 \quad 1 \quad 0.4) \begin{pmatrix} 1-0.10\pi_I \\ 0.05\pi_I \\ 0.05\pi_I \end{pmatrix}$$ giving $\pi_I = 0.30581$.

Similarly,

$$B_J(1,2) = \frac{1}{1+r_{01}} C^T q_J$$

$$\frac{1}{1.10} = \frac{1}{1.08}(1 \quad 1 \quad 0.4) \begin{pmatrix} 1-0.10\pi_J \\ 0.20\pi_{JI} \\ 0.10\pi_{JI} \end{pmatrix}$$ giving $\pi_J = 0.30303$.

Risky neutral transition matrix

$$Q(1) = \begin{matrix} I \\ J \\ D \end{matrix} \begin{pmatrix} 0.9694 & 0.0153 & 0.0153 \\ 0.0303 & 0.9394 & 0.0303 \\ 0 & 0 & 1.00 \end{pmatrix}$$

Duffie-Singleton model

Formulation in Duffie-Singleton model

- Treat default as an unpredictable event involving a sudden loss in market value.

- Default is assumed to occur at a risk-neutral hazard rate h_t; default over time Δt, given no default before time t, is approximately $h_t \Delta t$.

$$V_{risky} = \tilde{E}_0\left[\exp\left(-\int_0^T R_t dt\right)X\right]$$

where $\tilde{E}_t$ denotes risk-neutral expectation, R_t is the default-adjusted short-rate process, X is the value of contingent claim at maturity.

Default-adjusted short-rate process

$$R_t = r_t + h_t L_t + \ell_t$$

where r_t is the default-free short rate,

L_t is the *fractional loss* given default,

ℓ_t represents the *fractional carrying costs* of the defaultable claim (liquidity premium can be included),

h_t is the *arrival intensity* at time t (under risk neutral process) of a Poisson process whose first jump occurs at default.

Difficulties in parameter estimation

In order to disentangle the separate contributions of h and L , one would need additional data on

(i) default recovery values,

(ii) frequency of default of bonds of a given class.

- The exogeneity of h and L can misspecify some contractual features in some cases.

- The expected loss rate can switch from one regime to another in swap contracts depending on the reciprocal "moneyness" and "non-moneyness" of both counterparties through time.

Default probability density and hazard rate

τ^* – random variable representing the default time of a credit event

$Q(t) = P[\tau^* \leq t]$ = probability distribution function of τ^*

$$q(t) = \frac{dQ}{dt} = P[t < \tau^* < t + dt].$$

$q(t)$ is not the same as the hazard (default intensity) rate, $h(t)$.

Indeed, $h(t)\,dt$ is the probability of default between time t and $t + dt$ as seen as time t, assuming no default between time zero and time t.

$$h(t)t = P[t < \tau \le t + dt \mid \tau > t]$$

$$= \frac{Q(t+dt) - Q(t)}{1 - Q(t)} = \frac{q(t)dt}{1 - Q(t)}.$$

Define $G(t)$ = survival function = $P[\tau > t]$ = 1 - $Q(t)$,

then $h(t) = \frac{G^{'}(t)}{G(t)}$ and $G(0)$ =1.

Solving for $G(t)$, we obtain

$$G(t) = \exp\left(-\int_0^t h(s)\, ds \right).$$

Default probability density and hazard rate function are related by

$$q(t) = -G^{'}(t) = h(t)\exp\left(-\int_0^t h(s)\, ds \right).$$

Valuation of risky bonds

$v_i(t, T)$ = value of a defaultable zero-coupon bond of a firm that currently has credit rating i at time t, maturing at T

$$v_i(t, T) = P(t, T)[\phi + (1 - \phi) q_i(t, T)]$$

where ϕ = recovery ratio

$q_i(t, T)$ = probability of a default occuring after T, given that the debt has credit rating i as of time t.

$P(t, T)$ = value of a default-free zero-coupon bond

Madan-Unal model

Formulation of Madan-Unal model

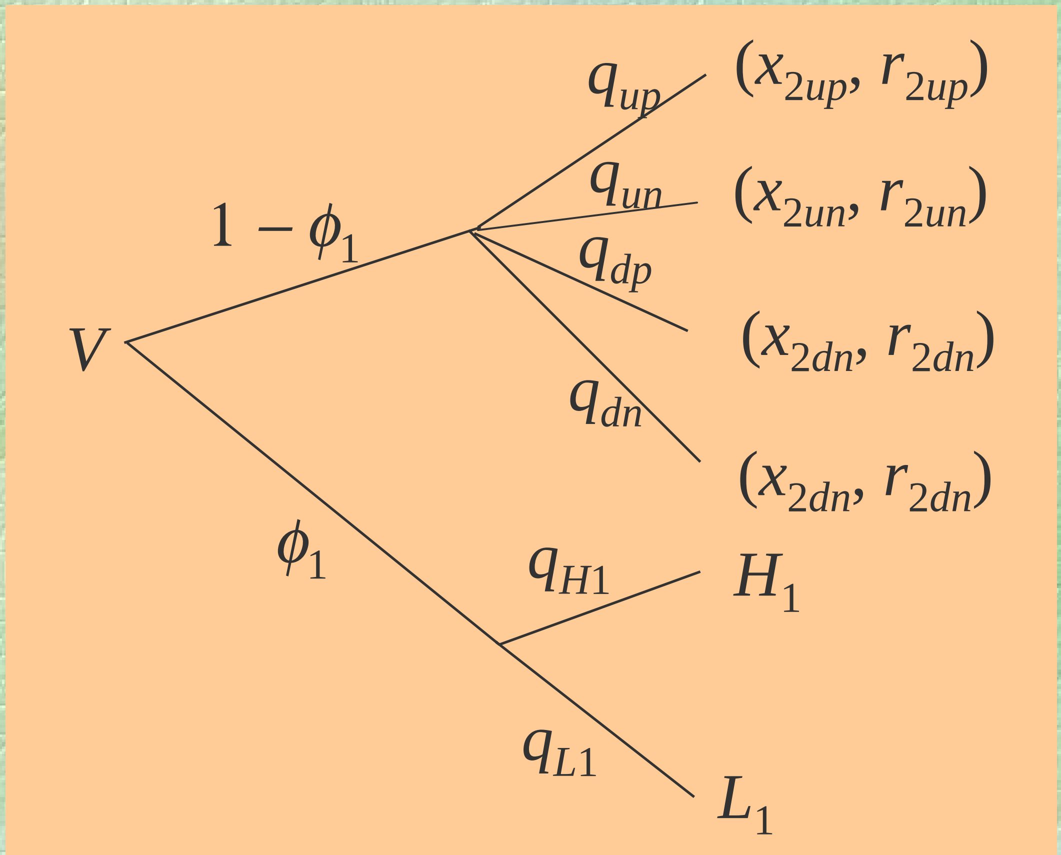
- Decomposes risky debt into two embedded securities:

– *survival security*: paying a dollar if there is no default and nothing otherwise

– *default security*: paying the rate of recovery in bankruptcy if default occurs and nothing otherwise

Evaluation of payoffs

- Default occurs with probability ϕ_1 .
- Recovery conditional on default is high (H_1) or low (L_1) with probabilities q_{H1} and q_{L1}.
- If there is no default in the first period, then second period outcomes depend on the evolution of firm specific information x and interest rates r.



Risk of recovery in default

Use the option components of junior and senior debt to extract information on the *default payout distribution* from the market prices of these debt instruments.

- Merton model – *predetermined* distribution given by the shortfall of firm value relative to the promised payment.

- Most other models assume a *constant* payout rate conditional on default.

Structural models require	*Reduced form models require*
1. Issuer's asset value process and issuer's capital structure	1. Issuer's default (bankruptcy) process
2. Loss given default; terms and conditions of the debt issue	2. Loss given default (can be specified as a stochastic process)
3. Default-risk-free interest rate process	3. Default-risk-free interest rate process
4. Correlation between the default-risk-free interest rate and the asset price	4. Correlation between the default-risk-free interest rate and the asset price

Different models measure the same risk but

- *impose different restrictions and distributional assumptions*
- *different techniques for calibration and solution.*