Reduced form models

General features of the reduced form models

- describe the process for the *arrival of default* – unpredictable event governed by an intensity-based or hazard rate process

- based on *contingent claims methodology*
  (adopt the term structure modeling technique commonly used for interest rate derivatives)

- avoid the problems associated with unobservable asset values and complex capital structures; e.g. when the issuer is a municipal government, then what "firm value" to use?
  (however, lack a structural definition of the default event)

- reliance on *credit spread data* to estimate the risk neutral probability of default

Reduced Form Models

$$\text{price of risky zero-coupon bond} = \text{present value factor} \times [(1 - q) \times \text{par value} + q \times \text{recovery}]$$

where $q = Q(\tau^* < T)$ risk neutral probability of default prior to maturity, present value factor is the price of riskfree zero-coupon bond.

* The time of default τ^* is assumed to follow a stochastic process governed by its own distribution (parameterized by a hazard rate process).

* The risk-neutral default probability (market-based) can be obtained as a function of the two discount factors (credit spread).

Jarrow-Turnbull model

Economy in Jarrow-Turnbull model

Two classes of zero-coupon bonds are traded:-

1. *Default-free, zero coupon bonds of all maturities*

   $P_0(t, T)$ denotes the time t dollar value of the default-free zero-coupon bond with unit par;

   $M(t)$ denotes the time t dollar value of the money market account initialized with one dollar at time 0.

2. *XYZ zero coupon risky bonds of all maturities*

   $v_1(t, T)$ denotes the time t dollar value of *XYZ* bond with unit par.

Assumptions

1. Constant recovery rate.

2. Default time is exponentially distributed with parameter λ.

3. Default-free rate process, hazard rate process and LGD function are mutually independent.

Foreign currency analogy

Dollar value of an *XYZ* bond is the XYZ value of the bond times the spot exchange rate dollar per *XYZ*, that is,

$$v_1(t, T) = P_1(t, T)e_1(t).$$

$P_1(t, T)$ is the default free XYZ bond price in XYZ currency world.

The XYZ bonds become default free in XYZ currency world. The pseudo spot exchange rate $e_1(t)$ is interpreted as the *payoff ratio in default*.

$$e_1(t) = \begin{cases} 1 & \text{if no default} \\ \text{recovery rate} & \text{if default occurs} \end{cases}.$$

Continuous framework of valuation of riskfree bonds

The money market account $M(t)$ accumulates at the spot rate $r(t)$

$$M(t) = \exp \int_0^t r(s)ds.$$

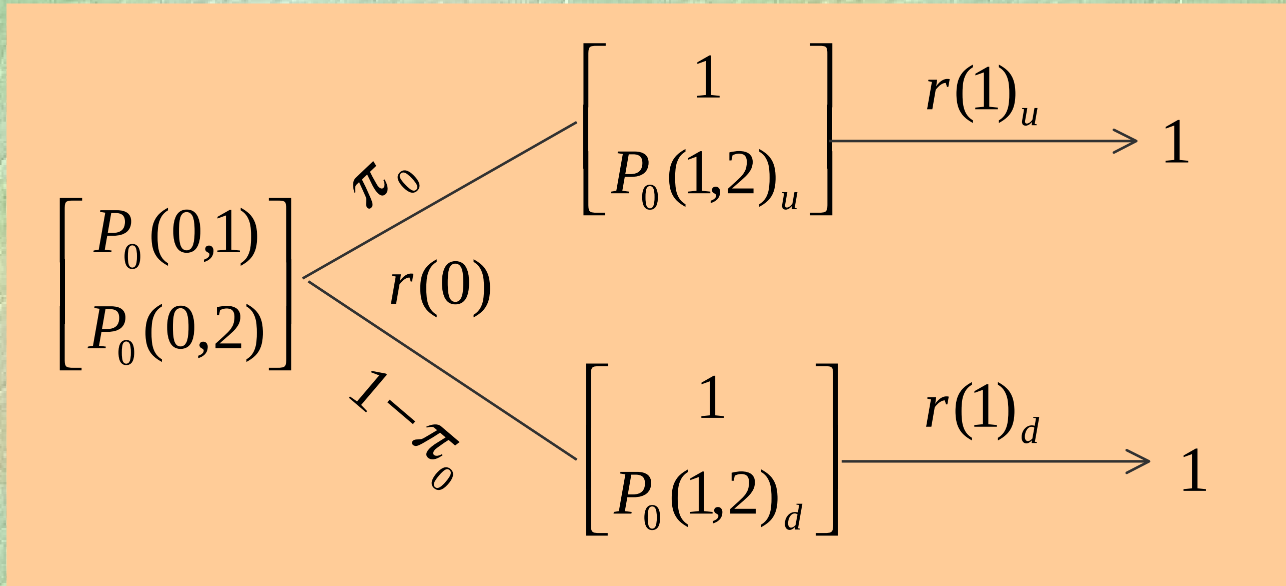
Under the assumption of *arbitrage free* and *complete market*, the default-free bond price $p_0(t, T)$ is given by

$$p_0(t,T) = \widetilde{E}_t\left(\frac{M(t)}{M(T)}\right)$$

where the expectation is taken under the unique equivalent martingale measure $\widetilde{Q}$.

Default free term structure

Bond price process for default-free debt is assumed to depend only on the spot interest rate.



$M(1) = r(0)$, $M(2)_u = r(0)r(1)_u$ and $M(2)_d = r(0)r(1)_d$;

$r(0) = 1/P_0(0,1)$, $r(1)_u = 1/P_0(1,2)_u$ and $r(1)_d = 1/P_0(1,2)_d$.

π_0 = risk-neutral probability of state u occurring (obtained from an assumed interest rate model)

Arbitrage-free restrictions

Non-existence of arbitrage opportunities is equivalent to the existence of pseudo probability π_0 such that $P_0(t, 1)/M(t)$ and $P_0(t, 2)/M(t)$ are martingales; market completeness is equivalent to uniqueness of these pseudo probabilities.

$$P_0(0,2) = \left[\pi_0 P_0(1,2)_u + (1-\pi_0)P_0(1,2)_d\right] \Big/ r(0)$$

giving

$$\pi_0 = \left[P_0(1,2)_d - r(0)P_0(0,2)\right] \Big/ [P_0(1,2)_d - P_0(1,2)_u].$$

Time 0 long-term zero-coupon bond price is the discounted expected value of time 1 bond prices using the pseudo probabilities.

π_0 exists, is unique, and satisfies $0 < \pi_0 < 1$ if and only if

$$P_0(1,2)_u < r(0)P_0(0,2) < P_0(1,2)_d.$$

Long-term zero-coupon bond should not be denominated by the short-term zero-coupon bond.

Remark

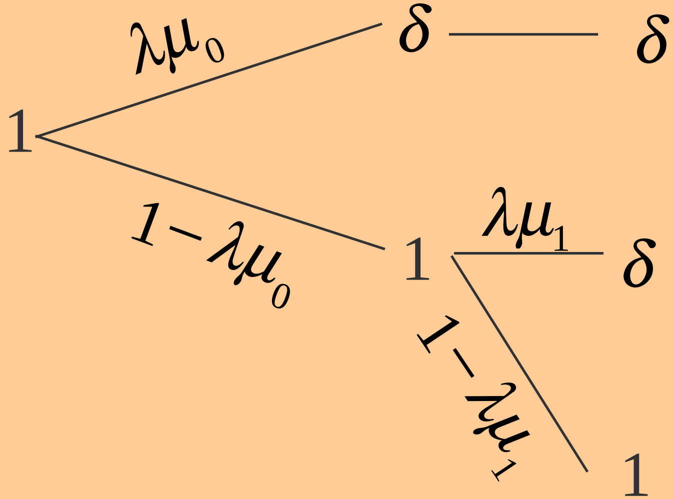
If $P_0(1, 2)_u < \mathrm{P}_0(1, 2)_d < r(0)P_0(0, 2)$, then we can arbitrage by shorting the bond, investing the proceed of $P_0(0, 2)$ in bank to earn $r(0)\mathrm{P}_0(0, 2)$.

Assumptions of the default process

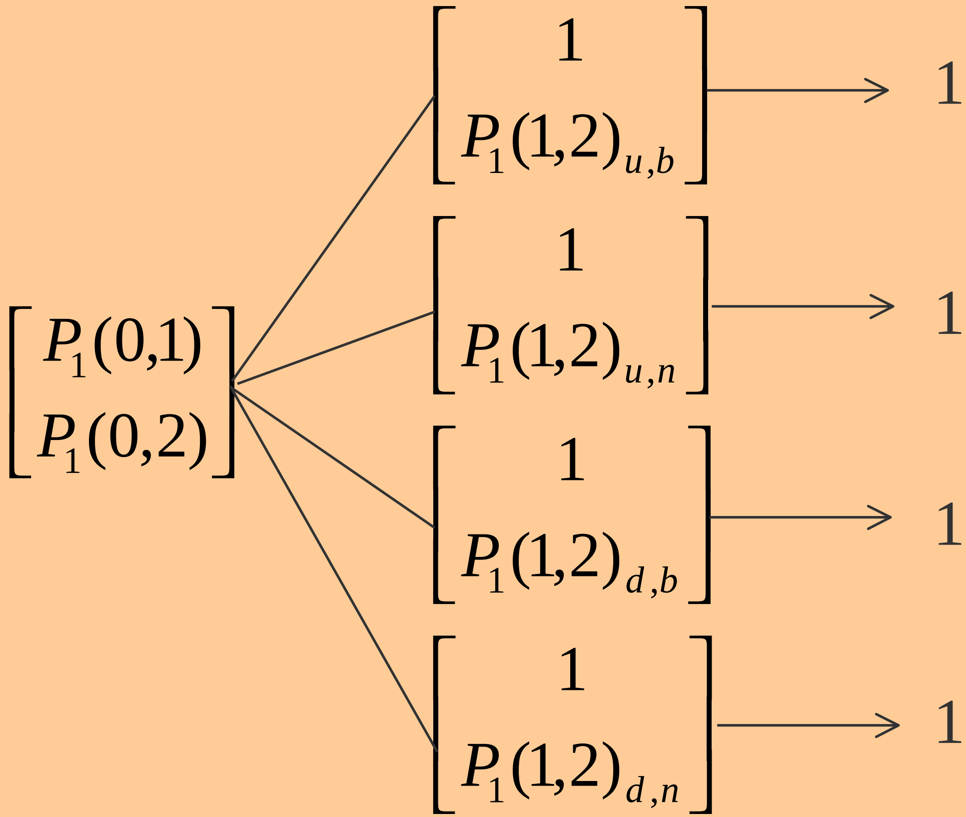
- Payoff to the bondholder in the event of default is taken to be an *exogenously given constant*, δ. It is assumed to be the same for all instruments in a given credit risk class.

- The spot interest rate process and the process of the arrival of default are *independent* under the pseudo probabilities.

Two-period discrete trading economy

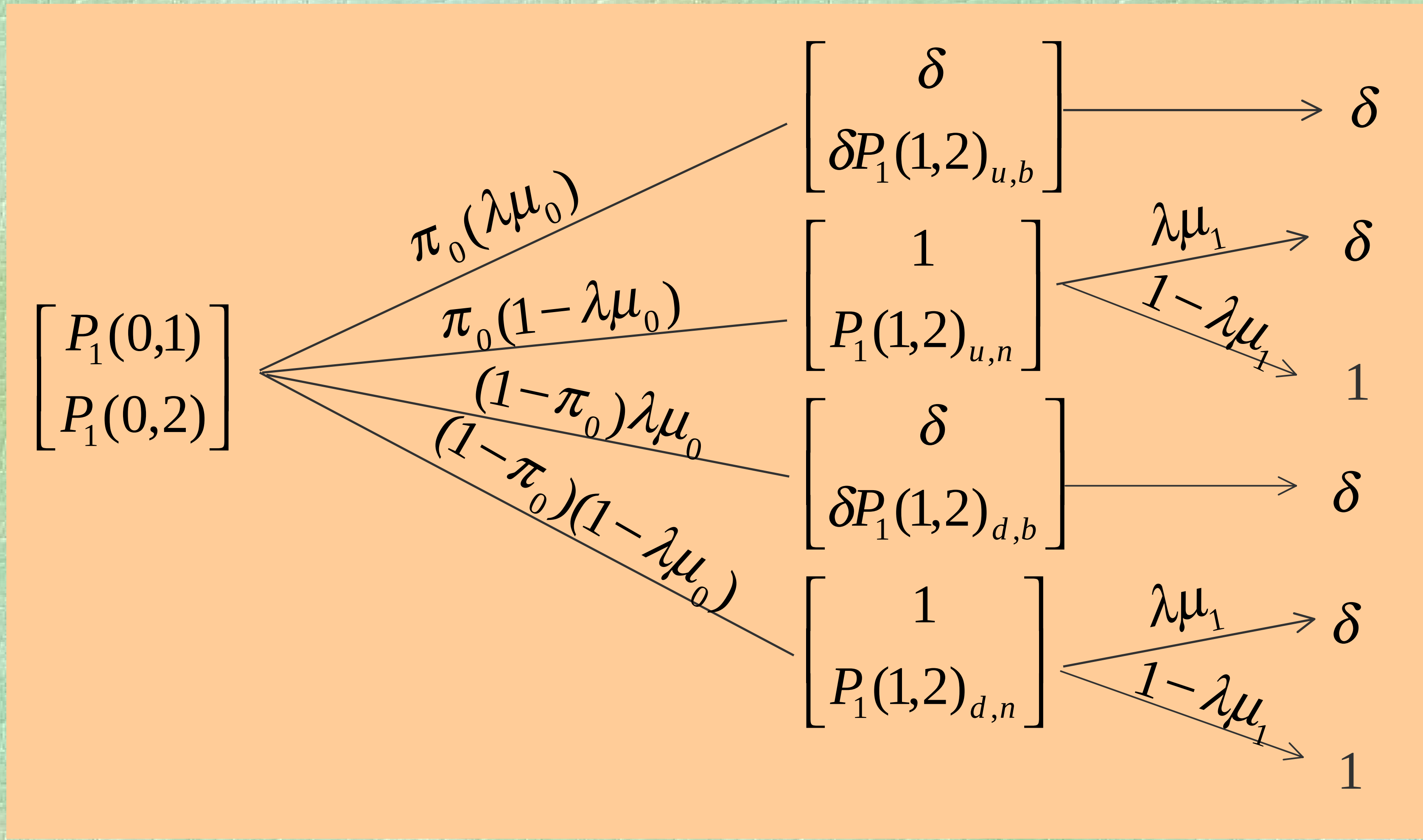


Payoff ratio process for XYZ debt



XYZ zero-coupon bond price process in XYZ currency world

XYZ zero-coupon bond price process in dollars



XYZ term structure I

By analyzing time 1 risky debt market

$$v_1(1,2)_{u,b} = \delta P_1(1,2)_{u,b} = \delta / r(1)_u$$
$$v_1(1,2)_{u,n} = P_1(1,2)_{u,n} = [\lambda\mu_1\delta + (1-\lambda\mu_1)] / r(1)_u$$
$$v_1(1,2)_{d,b} = \delta P_1(1,2)_{d,b} = \delta / r(1)_d$$
$$v_1(1,2)_{d,n} = P_1(1,2)_{d,n} = [\lambda\mu_1\delta + (1-\lambda\mu_1)] / r(1)_d$$

giving

$$\begin{aligned}\lambda\mu_1 &= \left[1 - P_1(1,2)_{u,n} r(1)_u\right] / (1-\delta) \\ &= \left[1 - P_1(1,2)_{d,n} r(1)_d\right] / (1-\delta).\end{aligned}$$

XYZ term structure II

By analyzing time 0 risky debt market

$$v_1(0,1) = P_1(0,1) = \left[\lambda\mu_0\delta + (1-\lambda\mu_0)\right] / r(0)$$

$$\begin{aligned} v_1(0,2) = P_1(0,2) = \Big[&\pi(\lambda\mu_0)\delta P_1(1,2)_{u,b} + \pi_0(1-\lambda\mu_0)P_1(1,2)_{u,n} \\ &+ (1-\pi_0)\lambda\mu_0\delta P_1(1,2)_{d,b} \\ &+ (1-\pi_0)(1-\lambda\mu_0)P_1(1,2)_{d,b}\Big] / \mathrm{r}(0), \end{aligned}$$

giving

$$\begin{aligned} \lambda\mu_0 = &\left[r(1)_d\, P_1(1,2)_{d,n} - P_1(0,2) / P_0(0,2)\right] / \\ &\left[r(1)_d\, P_1(1,2)_{d,n} - \delta\right] \end{aligned}$$

XYZ zero-coupon bonds

Under the pseudo probabilities, the expected payoff ratios at future dates are

$$\tilde{E}_1(e_1(2)) = \begin{cases} \delta & \text{if bankrupt at time 1} \\ \lambda\mu_1\delta + (1-\lambda\mu_1) & \text{if not bankrupt at time 1} \end{cases}$$

$$\tilde{E}_0(e_1(2)) = \lambda\mu_0\delta + (1-\lambda\mu_0)\left[\lambda\mu_1\delta + (1-\lambda\mu_1)\right]$$

$$\tilde{E}_0(e_1(1)) = \lambda\mu_0\delta + (1-\lambda\mu_0)$$

Decomposition:-

$$v_1(t,T) = P_0(t,T)\tilde{E}_t(e_1(T)).$$

Given observed bond prices $v_1(t, T)$ and $P_0(t, T)$, one can estimate $\tilde{E}_t(e_1(T))$.

Procedure (recursive estimation)

1. Given $P_0(0, 1)$ and $v_1(0, 1)$, estimate $\lambda\mu_0$ using
   $v_1(0, 1) = P_0(0, 1)\, \tilde{E}_0(e_1(1))$ and $\tilde{E}_0(e_1(1)) = \lambda\mu_0\delta + (1 - \lambda\mu_0)$.

- Given $P_0(0, 2)$ and $v_1(0, 2)$, estimate $\lambda\mu_1$ using

  $v_1(0, 2) = P_0(0, 2)\, \tilde{E}_0(e_1(2))$ and

  $\tilde{E}_0(e_1(2)) = \lambda\mu_0\delta + (1 - \lambda\mu_0)\, [\lambda\mu_1\delta + (1 - \lambda\mu_1)]$.

Numerical example

Take $\delta = 0.32$

maturity	$P_0(0, T)$	$V_0(0, T)$
1	94.8627	94.2176
2	89.5343	87.1168

The default-free spot interest rate process is determined by some interest rate model.

$r(1)_u$=6.359%, $P_0(1,2)_u$=0.9384

$r(0)$ = 5.274%

$\boxed{\pi_0 = 0.5}$

$r(1)_d$=5.206%, $P_0(1,2)_d$=0.9493

Using $v_1(0, 1) = P_0(0, 1)[\lambda\mu_0\delta + (1 - \lambda\mu_0)]$, we obtain

$$\lambda\mu_0 = 0.01.$$

From $v_1(0, 2) = P_0(0, 2)\ \{\lambda\mu_0\delta + (1 - \lambda\mu_0)[\lambda\mu_1\delta + (1 - \lambda\mu_1)]\}$, we obtain

$$\lambda\mu_1 = 0.03.$$

Option on a credit risky bond

European put option with maturity one year on a two-year *XYZ* zero-coupon bond. At option's maturity, option holder can sell the *XYZ* zero-coupon bond for the strike price of 92.

Let the face value of *XYZ* zero-coupon bond be 100.

Mathematical formulation

Put value at time 0

$$= P(0) = (1- \lambda\mu_0)\,[\pi_0\, P(1)_{u,\,n} + (1 - \pi_0)P(1)_{d,\,n}]/r(0)$$
$$+ \lambda\mu_0\,[\pi_0\, P(1)_{u,\,b} + (1 - \pi_0)P(1)_{d,\,b}]/r(0)$$

$$\begin{cases} v_1(1,2) = \delta P_0(1,2)_u = 0.3003 \\ P(1) = 61.97 \end{cases}$$

$$\begin{cases} v_1(1,2) = P_0(1,2)_u\,[\lambda\mu_1\delta + (1-\lambda\mu_1)] = 0.9193 \\ P(1) = 0.07 \end{cases}$$

$$\begin{cases} v_1(1,2) = \delta P_0(1,2)_d = 0.3038 \\ P(1) = 61.62 \end{cases}$$

$$\begin{cases} v_1(1,2) = P_0(1,2)_d\,[\lambda\mu_1\delta + (1-\lambda\mu_1)] = 0.9299 \\ P(1) = 0 \end{cases}$$

Probability of default = 0.01;

Probability of upward interest rate move = 0.5

$$P(0) = 0.9486\,[(1-0.01)(0.5\times 0.0) + 0.5\times 0)$$
$$+\,0.01(0.5\times 61.97 + 0.5\times 61.62)] = 0.62$$

Valuation of swap with counter-party risk

- Interest rate swap with two periods remaining (one period = 1 year).
- Fixed-rate payer (belongs to credit class XYZ) is paying 6% per annum.
- Floating-rate payer is considered default-free.
- The time 0 value of the two payments are

$$\text{FLOAT}(0, 1) = 1 - P_0(0, 1)$$
$$\text{FLOAT}(0, 2) = P_0(0, 1) - P_0(0, 2).$$

Bankruptcy rules

- If default occurs, all future payment are null and void.

The payoff ratio conditional upon no default at time $t - 1$

$$\overline{e}(t) = \begin{cases} 1 & \text{prob} = 1 - \lambda\mu_{t-1} \\ 0 & \text{prob} = \lambda\mu_{t-1} \end{cases}.$$

If default has occurred at $t - 1$, then

$$\overline{e}(t) = 0 \;\; \text{with prob} = 1.$$

Value of swap at t = 0 is

$$v_S(0) = \left[\overline{R}P_0(0,1) - \text{FLOAT}(0,1)\right]E_0\left[\overline{e}(1)\right] + \left[\overline{R}P_0(0,2) - \text{FLOAT}(0,2)\right]E_0\left[\overline{e}(2)\right]$$

where $\overline{R} =$ fixed payment = 6%.

Using the term structure given previously, with notational principal of \$100 million, we have

$$\begin{aligned} v_S(0) &= \{[0.06(0.9486) - (1 - 0.9486)]\,(1 - 0.01) \\ &\quad + [0.06(0.8953) - (0.9486 - 0.8953)] \\ &\quad + [1 - 0.03)(1 - 0.01)]\} \times 100 \text{ million} \\ &= 55{,}160(1 - 0.01) + 4{,}180(1 - 0.03)(1 - 0.01) = 58{,}622. \end{aligned}$$

If credit risk is ignored, then the value of swap becomes 59,340.