Optimal Shouting Policies of Options with Shouting Rights*

presented by

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The holders are allowed to **reset** certain terms of the derivative contract according to some *pre-specified rules*.

Terms that are resettable

- strike prices
- maturities

Voluntary (shouting) or *automatic* resets; other constraints on resets may apply.

Examples

1. S *& P 500 index bear market put warrants* with a 3-month reset (started to trade in 1996 in CBOE and NYSE)
   - original exercise price of the warrant, X
     = closing index level on issue date
   - exercise price is reset at the closing index level S_t on the reset date if $S_t > X$ (automatic reset)

Reset-strike warrants are available in Hong Kong and Taiwan markets.

2. *Reset feature in Japanese convertible bonds*

- reset downward on the conversion price

Sumitomo Banks 0.75% due 2001

Issue date	May 96
First reset date	31 May 1997
Annual reset date thereafter	31 May
Reset calculation period	20 business day period, excluding holidays in Japan, ending 15 trading days before the reset day
Calculation type	Simple average over calculation period

3. *Executive stock options*

- resetting the strike price and maturity

4. *Corporate debts*

- strong incentive for debtholders to extend the maturity of a defaulting debt if there are liquidation costs

5. *Canadian segregated fund*

- Guarantee on the return of the fund (protective floor); guarantee level is simply the strike price of the embedded put.

- Two opportunities to reset per year (at any time in the year) for 10 years. Multiple resets may involve sequentially reduced guarantee levels.

- Resets may require certain fees.

Objectives of our work

Examine the *optimal shouting policies* of options with *voluntary* reset rights.

Free boundary value problems

- Critical asset price level to shout;
- Characterization of the optimal shouting boundary for one-shout and multi-shout models (analytic formulas, numerical calculations and theoretical analyses)

Resettable put option

The strike price is reset to be the prevailing asset price at the shouting moment chosen by the holder.

$$\text{Terminal payoff} = \begin{cases} \max(X - S_T, 0) & \text{if no shout occurs} \\ \max(S_t - S_T, 0) & \text{if shouting occurs at time } t \end{cases}$$

Shout call option

$$\text{Terminal payoff} = \begin{cases} \max(S_T - X, S_t - X) & \text{if shouting occurs at time } t,\ S_t > X \\ \max(S_T - X, 0) & \text{if no shout occurs} \end{cases}$$

Shout floor (protective floor is not set at inception)

Shout to install a protective floor on the return of the asset.

$$\text{Terminal payoff} = \begin{cases} \max(S_t - S_T, 0) & \text{if shouting occurs at time } t \\ 0 & \text{if no shout occurs} \end{cases}$$

Relation between the resettable put option and the shout call option

Since $\begin{cases} \max(S_T - X, S_t - X) \\ \max(S_T - X, 0) \end{cases} = (S_T - X) + \begin{cases} \max(X - S_T, 0) \\ \max(S_t - S_T, 0) \end{cases},$

so

price of one-shout shout-call option
= price of one-shout resettable put option + forward contract.

- Both options share the same optimal shouting policy.
- Same conclusion applied to multi-shout options.

Formulation as free boundary value problems

- Both one-shout resettable put option and one-shout shout floor become an at-the-money put option upon shouting.
- Price function of an at-the-money put option is $SP_1(\tau)$, where

$$P_1(\tau) = e^{-r\tau} N(-d_2) - e^{-q\tau} N(-d_1)$$

where $$d_1 = \frac{r - q + \frac{\sigma^2}{2}}{\sigma}\sqrt{\tau} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{\tau}.$$

Linear complementarity formulation of the pricing function $V(S, \tau)$

$$\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r - q) S \frac{\partial V}{\partial \tau} + rV \geq 0, \qquad V(S, \tau) \geq SP_1(\tau),$$

$$\left[\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r - q) S \frac{\partial V}{\partial S} + rV\right]\left[V - SP_1(\tau)\right] = 0$$

$$V(S,0) = \begin{cases} \max(X - S, 0) & \text{for resettable put} \\ 0 & \text{for shout floor} \end{cases}.$$

Properties of $P_1(\tau)$

(i) If $r \le q$, then

$$\frac{d}{d\tau}\left[e^{q\tau}P_1(\tau)\right] > 0 \qquad \text{for} \qquad \tau \in (0,\infty).$$

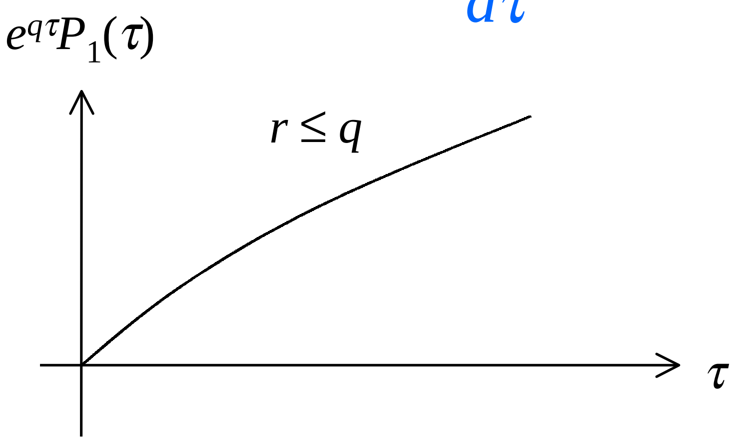
(ii) If $r > q$, then there exists a unique critical value $\tau_1^* \in (0,\infty)$ such that

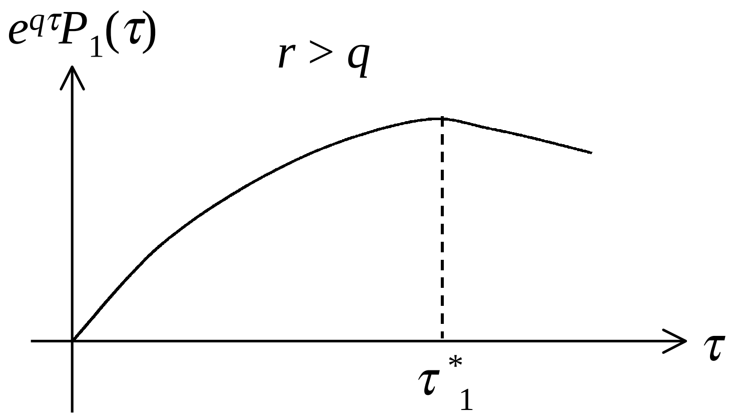
$$\frac{d}{d\tau}\left[e^{q\tau}P_1(\tau)\right]\Big|_{\tau=\tau_1^*} = 0,$$

and

$$\frac{d}{d\tau}\left[e^{q\tau}P_1(\tau)\right] > 0 \quad \text{for} \quad \tau \in (0,\tau_1^*)$$

$$\frac{d}{d\tau}\left[e^{q\tau}P_1(\tau)\right] < 0 \quad \text{for} \quad \tau \in (\tau_1^*,\infty)$$





Price of the one-shout shout floor, $R_1(S, \tau)$

$$R_1(S,\tau) = S\ g(\tau)$$

Substituting into the linear complementarity formulation:

$$\frac{d}{d\tau}\left[e^{q\tau}q(\tau)\right] \geq 0, \qquad g(\tau) \geq P_1(\tau)$$

$$\frac{d}{d\tau}\left[e^{q\tau}g(\tau)\right]\left[g(\tau) - P_1(\tau)\right] = 0$$

$$g(0) = 0.$$

(i) $r \leq q$

$$\frac{d}{d\tau}\left[e^{q\tau}P_1(\tau)\right] > 0 \quad \text{for} \quad \tau > 0 \ \text{ and } \ P_1(0) = 0;$$

so

$$g(\tau) = P_1(\tau), \qquad \tau \in (0, \infty).$$

(ii) $r > q$

$$g(\tau) = P_1(\tau) \quad \text{for} \quad \tau \in (0, \tau_1^*].$$

When $\tau > \tau^*_1$, we cannot have $g(\tau) = P_1(\tau)$ since this leads to

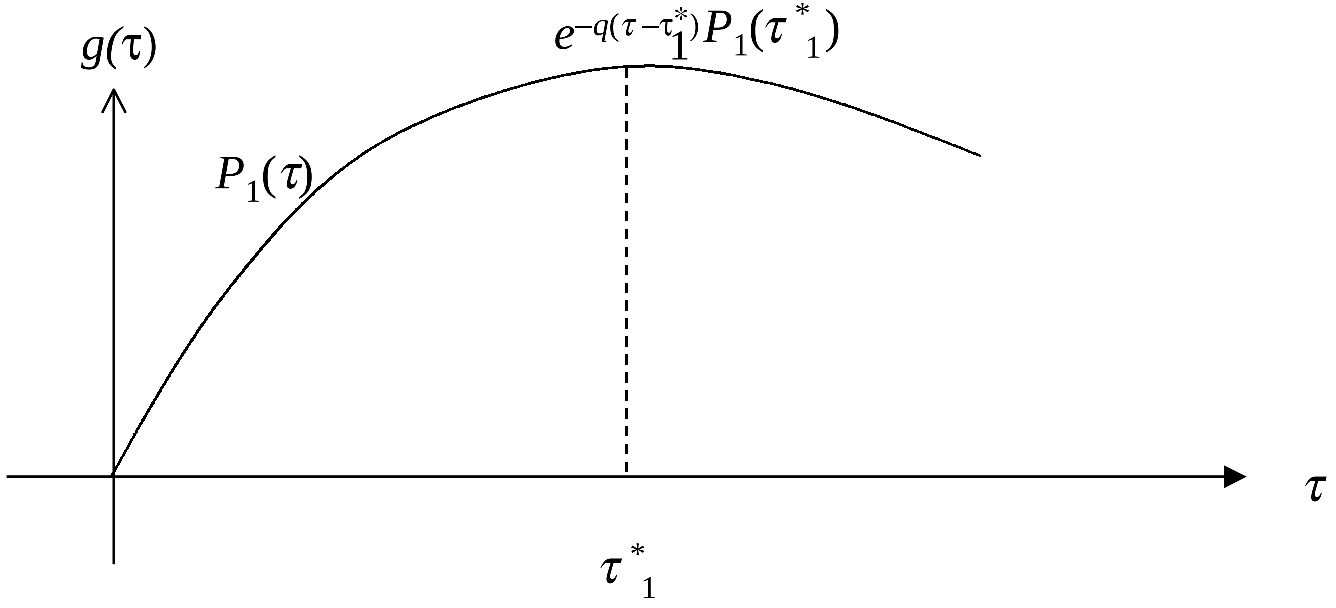
$$\frac{d}{d\tau}\left[e^{q\tau} g(\tau)\right] = \frac{d}{d\tau}\left[e^{q\tau} - P_1(\tau)\right] = 0,$$

a contradiction. Hence,

$$\frac{d}{d\tau}\left[e^{q\tau} g(\tau)\right] = 0 \quad \text{for} \quad \tau \in (\tau_1^*, \infty).$$

Solving

$$g(\tau) = e^{-q(\tau-\tau_1^*)} P_1(\tau_1^*) \quad \text{for} \quad \tau \in (\tau_1^*, \infty).$$



Optimal shouting policy of the shout floor

- does not depend on the asset price level (due to linear homogeneity in S)

- when $\frac{d}{d\tau}\left[e^{q\tau}P_1(\tau)\right] \geq 0, R_1(S,\tau) = SP_1(\tau)$,
  inferring that the holder should shout at once.

Summary

(i) $r \leq q,\ \tau \in (0, \infty)$ or (ii) $r > q,\ \tau \leq \tau_1^*$

holder should shout at once

- $r > q,\ \tau \leq \tau_1^*$

  holder should not shout at any asset price level.

Optimal shouting boundary for the resettable put option

Asymptotic behavior of $S_1^*(\tau)$ *as* $\tau \to 0^+$

$$S_1^*(0^+) = \text{X, independent of the ratio of } r \text{ and } q.$$

Proof $\quad D_1(S,\tau) = V_1(S,\tau) - SP_1(\tau)$

$$\frac{\partial D_1}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 D_1}{\partial S^2} - (r-q)S\frac{\partial D_1}{\partial S} + rD_1 = -S\left[P_1'(\tau) + qP_1(\tau)\right]$$

$$0 < S < S_1^*(\tau),\ \tau > 0$$

$$D_1(0,\tau) = Xe^{-r\tau} \text{ and } P_1(S_1^*(\tau),\tau) = 0$$

$$\frac{\partial D_1}{\partial S}(S_1^*(\tau),\tau) = 0, \quad D_1(S,0) = \max(X-S,0).$$

Note that $D_1(S,\tau) \geq 0$ for all τ; $-S[P_1'(\tau) + qP_1(\tau)) \to -\infty$ as $\tau \to 0^+$.
Assuming $S_1^*(0^+) > X$, then for $S \in (X, S_1^*(0^+))$, we have

$$\frac{\partial D_1}{\partial \tau}(S,0^+) = -S\left[P_1'(\tau) + qP_1(0)\right] < 0, \quad \text{a contradiction.}$$

Financial intuition dictates that $S_1^*(0^+) \geq X$.

Asymptotic behaviour of $S_1^*(\tau)$ *as* $\tau \to \infty$

$S^*_{1,\infty}$ exists when r < q; this is linked with the existence of the limit:

$$\lim_{\tau\to\infty} e^{e\tau} P_1(\tau) = 1 \quad \text{for } r < q.$$

Write $W_1(S,\tau) = e^{r\tau} V(S,\tau);\quad W_1^\infty(S) = \lim_{\tau\to\infty} W_1(S,\tau).$

$$\frac{\sigma^2}{2} S^2 \frac{d^2 W_1^\infty}{dS^2} + (r-q)S\frac{dW_1^\infty}{dS} = 0,\quad 0 < S < S^*_{1,\infty}$$

$$W_1^\infty(0) = X,\quad W_1^\infty(S^*_{1,\infty}) = S^*_{1,\infty},$$

$$\frac{dW_1^\infty}{dS}(S^*_{1,\infty}) = 1.$$

$$W_1^\infty(S) = X + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}} X^{-\alpha} S^{1+\alpha},\qquad 0 < S < S^*_{1,\infty},$$

where $S^*_{1.\infty} = \left(1 + \frac{1}{\alpha}\right)X$ and $\alpha = \frac{2(q-r)}{\sigma^2}$ (α becomes zero when $r = q$).

When $r > q$, $V_1(S, \tau) \geq R_1(S, \tau) > SP_1(\tau)$ for $\tau > \tau_1^*$.
It is never optimal to shout at $\tau > \tau_1^*$.

Lemma For $r > q$ and $\tau_0 < \tau_1^*$, there exists a critical asset price $S_1^*(\tau_0)$ such that $V_1(S, \tau_0) = SP_1(\tau_0)$ for $S_1 \geq S^*(\tau_0)$.

Integral equation for $S_1^*(\tau)$

$$S_1^*(\tau)P_1(\tau) = P_E(S_1^*(\tau),\tau) + S_1^*(\tau)e^{-q\tau}\int_0^\tau N(d_{1.\tau-u}^*)\frac{d}{du}[e^{qu}P_1(u)]\,du$$

where

$$d_{1,\tau-u}^* = \frac{\ln\frac{S_1^*(\tau)}{S_1^*(u)} + \left(r - q + \frac{\sigma^2}{2}\right)(\tau - u)}{\sigma\sqrt{\tau - u}}.$$

Pricing formulation of the *n*-shout resettable put option

Terminal payoff $= \max(S_{t_\ell} - S_{T,0})$, where t_ℓ is the last shouting instant, $0 \le \ell \le n$.

Define $P_n(\tau) = V_{n-1}(1, \tau; X = 1)$, then

$$\frac{\partial V_n}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V_n}{\partial S^2} - (r-q) S \frac{\partial V_n}{\partial S} + r V_n \ge 0, \qquad V_n(S,\tau) \ge S P_n(\tau),$$

$$\left[\frac{\partial V_n}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V_n}{\partial S^2} - (r-q) S \frac{\partial V_n}{\partial S} + r V\right]\left[V_n - S P_n(\tau)\right] = 0$$

$$V_n(S,0) = \max(X - S, 0).$$

* The analytic form for $P_n(\tau)$, $n > 1$.

Properties of the price function and optimal shouting boundaries

(i) $r < q$, $S_n^*(\tau)$ is defined for $\tau \in (0, \infty)$

$$S_{n+1}^*(\tau) < S_n^*(\tau), \qquad n = 1,2$$
$$S_n^*(0^+) = X$$

$S_n^*(\tau)$ is an increasing function of τ with a finite asymptotic value as $\tau \to \infty$.

$$S_{2,\infty}^* = \frac{1+\frac{1}{\alpha}}{1+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}} \text{ and } S_{3,\infty}^* = \frac{1+\frac{1}{\alpha}}{1+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\left[1+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\right]}.$$

(ii) $r < q$, $S_n^*(\tau)$ is defined only for $\tau \in (0, \tau_n^*)$, where τ_n^* is given by the solution to

$$\frac{d}{d\tau}\left[e^{q\tau} P_n(\tau)\right] = 0.$$

Summaries and conclusions

- The behaviors of the optimal shouting boundaries of the resettable put options depends on $r > q$, $r = q$ or $r < q$.

- Monotonic properties
  (i) an option with more shouting rights outstanding should have higher value;
  (ii) the holder shouts at a lower critical asset price with more shouting rights outstanding;
  (iii) the holder chooses to shout at a lower critical asset price for a shorter-lived option;
  (iv) the critical time earlier than which it is never optimal to shout increases with more shouting rights outstanding.
- Analytic price formula of the one-shout shout floor and integral representation of the shouting premium of the one-shout resettable put are obtained.