1. Dynamic risk models

Credit risk consists of two components: default risk and spread risk

1. Default risk: any non-compliance with the exact specification of a contract.
2. Spread risk: reduction in market value of the contract / instrument due to changes in the credit quality of the debtor / counterparty.

   – price or yield change of a bond as a result of credit rating downgrade

Event of default

1. Arrival risk – timing of the event
2. Magnitude risk – loss amount / recovery value

Risk elements

1. Exposure at default / recovery rates – both are random variables
2. Default probability
3. Transition probabilities – the process of changing the creditworthiness is called credit migration.

Bonds – securitized versions of loans and tradeable

Payment structure

1. Upfront payment by the investor to purchase the bond.
2. On the coupon dates, the investor receives coupon (fixed or floating) from the issuer.
3. On bonds maturity date, the issuer pays the par value and the final coupon payment.

Bundles of risk embedded

- duration and convexity (sensitivity to the interest rate movement)
- credit risk: risk of default and risk of volatility in credit spreads
- early termination due to recall by issuer
- liquidity

Static hedge versus dynamic hedge: How to manage duration, convexity and callability risk independent of the bond position?

Pricing of credit derivatives and rating of credit linked notes whose payoff depends on certain credit event.

- Unambiguous definition of the credit event – bankruptcy, downgrade, restructuring, merger, payment default, etc.
- Possibility of default – default probability and hazard rate.
- Recovery value and settlement risk.
- Correlation of defaults between obligors / risky assets.

Remarks

1. The variability of default risk within a loan portfolio can be substantial. The highest default probability is significantly larger than the smallest default probability.

2. The correlation between default risks of different borrowers is generally low (that is, low joint default frequency), though it can be significant for related companies, and smaller companies within the same domestic industry sector.

3. The lack of correlation would increase the difficulty of hedging portfolio default risk with tradable instruments.

4. The best resort to reduce default risk is diversification.

Modeling default risk

Cumulative risk of default

This measures the total default probability of an obligor over the term of the obligation.

Some basic techniques

- Credit ratings, if the companies have been rated.
- Calculate key accounting ratios using the firm's financial data, then compared with the comparable median for rated firms – allow a rating equivalent to be determined.
- KMV model – based on stock price dynamics (for listed companies).

Credit spread: compensate investor for the risk of default on the underlying securities

$$\text{spread} = \text{yield on the loan} - \text{riskfree yield}$$

Construction of a credit risk adjusted yield curve is hindered by

1. The absence in money markets of liquid traded instruments on credit spread.

2. The absence of a complete term structure of credit spreads. At best we only have infrequent data points.

Term structure of credit spreads

The price of a corporate bond must reflect not only the spot rates for default-free bonds but also a risk premium to reflect default risk and any options embedded in the issue.

Simple approach

1. Take the spot rates that are used to discount the cash flows of corporate bonds to be the Treasury sport rates plus a constant credit spread.

2. Since the credit spread is expected to increase with maturity, we need *a term structure for credit spreads*.

Unlike Treasury securities, there are no issuers that offer a sufficiently wide range of corporate zero-coupon securities to construct a zero-coupon spread curve.

Type	*Maturity*	*Price per \$1 par*	*Yield(%)*
Treasury	1 year	0.930	7.39
Corporate	1 year	0.926	7.84
Treasury	2 years	0.848	8.42
Corporate	2 years	0.840	8.91

- For the one-year corporate security, the 4-cent difference produces a credit spread of 45 basis points.

- Price of corporate zero = Price of Treasury zero $\times$ (1$-$ probability of default)

- probability of default of one-year security

$$= 1 - \frac{0.926}{0.930} \approx 0.0043$$

probability of default of two-year security

$$= 1 - \frac{0.848}{0.840} \approx 0.0095.$$

- Forward probability of default: conditional probability of default in the second year, given that the corporation does not default in the first year.

Survival function

$T =$ continuous random variable that measures the default time

$$F(t) = P[T \leq t], \quad t \geq 0$$

Survival function $= S(t) = 1 - F(t) = P[T > t]$

$$\begin{aligned} \text{density function} \quad &= \quad f(t) = F'(t) = -S'(t) \\ &= \quad \lim_{\Delta \to 0} \frac{P[t \leq T < t + \Delta]}{\Delta}. \end{aligned}$$

$$\begin{aligned} {}_tq_x \quad &= \quad \text{conditional probability that risky security will default} \\ & \text{within the next } t \text{ years conditional on its survival for } x \text{ year} \\ &= \quad P[T - x \leq t | T > x], \quad t \geq 0 \\ {}_tp_x \quad &= \quad 1 - {}_tq_x, \quad t \geq 0 \\ S(t) \quad &= \quad {}_tp_0. \end{aligned}$$

For $t = 1$, we write

$$
\begin{aligned}
p_x &= P[T - x > 1 | T > x] \\
q_x &= P[T - x \leq 1 | T > x] \\
&= \text{marginal default probability} \\
&= \text{probability of default in the next year conditional on} \\
&\quad\ \text{the survival until the beginning of the year}
\end{aligned}
$$

A credit curve is simply the sequence of $q_0, q_1, \cdots, q_n$ in discrete models.

Hazard rate function

Gives the instantaneous default probability for a security that has survived up to time x

$$h(x)\Delta x = \frac{f(x)}{1-F(x)}\Delta x \approx \frac{F(x+\Delta x)-F(x)}{1-F(x)} = P[x < T \leq x{+}\Delta x | T > x]$$

so that

$$\begin{aligned} h(x) &= -\frac{S'(x)}{S(x)}, \text{ giving } S(t) = e^{-\int_0^t h(s)\,ds}. \\ t q_x &= e^{-\int_0^t h(s+x)\,ds} \\ {}_t q_x &= 1 - e^{-\int_0^t h(s+x)\,ds}. \end{aligned}$$

Also, $F(t) = 1 - S(t) = 1 - e^{-\int_0^t h(s)\,ds}$ and $f(t) = S(t)h(t)$.

When the hazard rate is constant, then

$$f(t) = he^{-ht}.$$

Reduced form approach

Default is modeled as a *point process*. We are not interested in the event itself but the sequence of random times at which the events occur.

- Over $[t, t + \Delta t]$ in the future, the probability of default, conditional on no default prior to time t, is given by $h_t \Delta t$, where h_t is referred to as the hazard rate process.

- Let Γ denote the time of default Conditional probability of default over $[t, t + \Delta t]$, given survival up to time t, is

$$P_r[t < \Gamma \leq t + \Delta t | \Gamma > t] = h_t \Delta t.$$

Comparison between reduced form model and structural model

This is an alternative to Merton's structural model. In Merton's model, the default occurs when the value of the firm falls below a pre-specified deterministic threshold (liabilities of the firm). In this case, the default time is then *predictable*.

- The default occurs as a complete surprise. It allows to add some randomness to the default threshold.

- It loses the micro-economic interpretation of the default time (the model comes from the reliability theory), but traders do not care for the purpose of pricing.

- Survival probability

$$P_r[\Gamma > t] = \exp\left(-\int_0^t h_s\, ds\right).$$

- Suppose we write $P_r[t < \Gamma \leq t + \Delta t] = f_t \Delta t$, where f_t is the density function of the default time, we then have

$$f_t = h_t \exp\left(-\int_0^t h_s\, ds\right).$$

- Probability of surviving until time t, given survival up to $s \leq t$,

$$P_r\left[\Gamma > t | \Gamma > s\right] = \frac{P_r[\Gamma > t]}{P_r[\Gamma > s]} = \exp\left(-\int_s^t h_s\, du\right).$$

- Default in $(t, t + \Delta t]$, conditional on no default up to time s,

$$P_r[t < \Gamma \leq t + \Delta t | \Gamma > s] = h_t \exp\left(-\int_s^t h_u\, du\right) \Delta t.$$

Suppose h_t is a random process with dependence on the history of a vector of macro-economic/firm specific random variables. Then

$$P_r[\Gamma > t] = E\left[\exp\left(-\int_0^t h_s\, ds\right)\right]$$

where the expectation is taken over all possible paths of the Brownian process.

Information set

Let G_t denote the information set or filtration such that h_t is a process adapted to G_t. Also, we let I_t denote the information set which tells whether default has occurred. The union of G_t and I_t is the full information $\mathcal{F}_t$ that contains information about the path history of the state variable process and the default history.

Value of credit-risky discount bond

$$
\begin{aligned}
B_t &= \text{value of a unit initialized money market account} \\
&= \exp\left(\int_0^t r_s\,ds\right) \quad \text{[without default risk],}
\end{aligned}
$$

where r_s is in general stochastic.

$$
\begin{aligned}
Q_{ab} &= \text{value at time } a \text{ of a credit-risky discount bond} \\
&\ \text{that matures at time } b \\
&= E_a\left[\exp\left(-\int_a^b r_s\,ds\right)\mathbf{1}_{\{\Gamma > b\}}\right] \\
&= B_a E_a\left[\frac{\mathbf{1}_{\{\Gamma > b\}}}{B_b}\right]
\end{aligned}
$$

where E_a denotes the conditional expectation (in risk neutral measure) given the full information set F_a up to time a. We need to specify the process that drives the occurrence of default.

Credit valuation model

1. Credit risk should be measured in terms of *probabilities and mathematical expectations*.

2. Credit risk model should be based on current, rather than historical measurements. The relevant variables are the *actual market values rather than accounting values*. It should reflect the development in the borrower's credit standing through time.

3. An assessment of the future earning power of the firm, company's operations, projection of cash flows, etc., has already been made by the aggregate of the market participants in the stock market. The challenge is how to *interpret the changing share prices* properly.

4. The various liabilities of a firm are *claims on the firm's value*, which often take the form of options, so the credit model should be consistent with the theory of option pricing.

Merton's firm value model

- Built upon a stochastic process of the firm's value.
- Aim to provide a link between the prices of equity and all debt instruments issued by one particular firm
- Default occurs when the firm value falls to a low level such that the issuer cannot meet the par payment at maturity or coupon payments.

Potential applications

1. Relative value trading between shares and debt of one particular issuer.
2. Default risk assessment of a firm based upon its share price and fundamental (balance sheet) data.

Assumptions in the Merton model

1. The firm asset value V_t evolves according to

$$\frac{dV}{V} = \mu\, dt + \sigma\, dZ$$

   $\mu =$ instantaneous expected rate of return.

2. The liabilities of the firm consist only of a single debt with face value F. The debt is assumed not to have coupon nor embedded option feature.
3. The debt is viewed as a contingent claim on the firm's asset. Default can be triggered only at maturity and this occurs when $V_T < F$.
4. Upon default, the firm is liquidated at zero cost and all the proceeds from liquidation are transferred to the debt holder.

The terminal payoff of this contingent claim can be expressed as

$$\min(V_T, F) = F - \underbrace{(F - V_T)^+}_{\text{put payoff}}, \qquad x^+ = \max(x, 0).$$

The debt holders have sold a put to the issuer – right to put the firm assets at the price of face value F.

Let $\overline{B}(V,t)$ denote the value of the risky debt at time t, then

$$\frac{\partial \overline{B}}{\partial t} + \frac{\sigma^2}{2} V^2 \frac{\partial^2 \overline{B}}{\partial V^2} + rV \frac{\partial \overline{B}}{\partial V} - r\overline{B} = 0,$$
$$\overline{B}(V,T) = F - \max(F - V_T, 0).$$

The solution is given by

$$\overline{B}(V,\tau) = Fe^{-r\tau} - P_E(V,\tau), \quad \tau = T - t$$
$$P_E(V,\tau) = Fe^{-r\tau} N(-d_2) - VN(-d_1) = \text{ expected loss}$$
$$d_1 = \frac{\ln \frac{V}{F} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}.$$

Equity value $E(V,\tau) = V - \overline{B}(V,\tau)$

$$= V - [Fe^{-r\tau} - P_E(V,\tau)] = C_E(V,\tau),$$

by virtue of the put-call parity relation. The result is not surprising since the shareholders have the contingent claim $\max(V_T - F, 0)$ at maturity.

Write the expected loss as

$$N(-d_2)\left[Fe^{-r(T-t)} - \frac{N(-d_1)}{N(-d_2)}V\right],$$

where $\dfrac{N(-d_1)}{N(-d_2)}$ is the *expected discounted recovery rate*.

$\overline{B}$ = present value of par $-$ default probability $\times$ expected discounted loss given default

where

$$\text{default probability } = N(-d_2) = P_r[V_T \leq F].$$

Numerical example

Data

$V_t = 100, \sigma_V = 40\%, \ell_t =$ quasi-debt-leverage ratio $= 60\%$,

$T - t = 1$ year and $r = \ln(1 + 5\%)$.

Calculations

1. Given $\ell_t = \dfrac{Fe^{-r(T-t)}}{V} = 0.6$,

   then $F = 100 \times 0.6 \times (1 + 5\%) = 63$.

2. Discounted expected recovery value

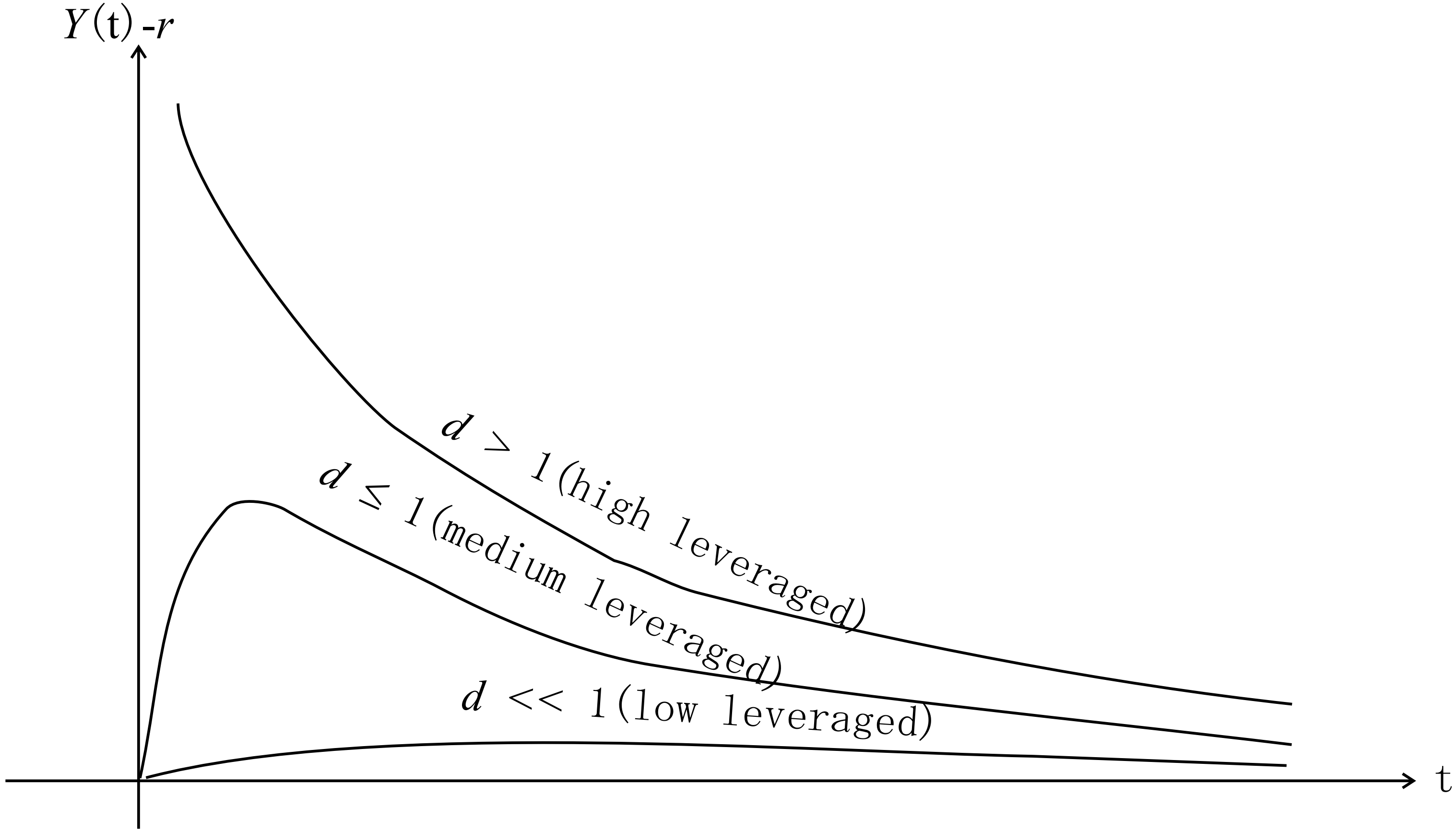
$$= \frac{N(-d_1)}{N(-d_2)}V = \frac{0.069829}{0.140726} \times 100 = 49.62.$$

3. Expected discounted shortfall amounts $= 63 - 49.62 = 10.38$.

4. Cost of default = put value

$$\begin{aligned} &= N(-d_2)\times \text{ expected discounted shortfall} \\ &= 14.07\% \times 10.38 = 1.46; \end{aligned}$$

value of credit risky bond is given by $60 - 1.46 = 58.54$.



- As time approaches maturity, the credit spread always tends to zero when $d \leq 1$ but tends toward infinity when $d > 1$.

- At times far from maturity, the credit spread has low value for all values of d since sufficient time has been allowed for the firm value to have a higher potential to grow beyond F.

Time dependent behaviors of credit spreads

- Downward-sloping for highly leveraged firms.
- Humped shape for medium leveraged firms.
- Upward-sloping for low leveraged firms.

Possible explanation

- For high-quality bonds, credit spreads widen as maturity increases since the upside potential is limited and the downside risk is substantial.

Remark

Most banking regulations do not recognize the term structure of credit spreads. When allocating capital to cover potential defaults and credit downgrades, a one-year risky bond is treated the same as a ten-year counterpart.

Industrial implementation: mKMV model

- *Expected default frequency* (EDF) is a forward-looking measure of actual probability of default. EDF is firm specific.

- KMV model is based on the structural approach to calculate EDF (credit risk is driven by the firm value process).

  - It is best when applied to publicly traded companies, where the value of equity is determined by the stock market.
  - The market information contained in the firm′s stock price and balance sheet are translated into an implied risk of default.

- Accurate and timely information from the equity market provides a continuous credit monitoring process that is difficult and expensive to duplicate using traditional credit analysis.

- Annual reviews and other traditional credit processes cannot maintain the same degree of "on guard" that EDFs calculated on a monthly or a daily basis can provide.

Key features in KMV model

1. Distance to default ratio determines the level of default risk.

   - This key ratio compares the firms net worth $E(V_T) - d^*$ to its volatility.
   - The net worth is based on values from the equity market, so it is both timely and superior estimate of the firm value.

2. Ability to adjust to the credit cycle and ability to quickly reflect any deterioration in credit quality.

3. Work best in highly efficient liquid market conditions.

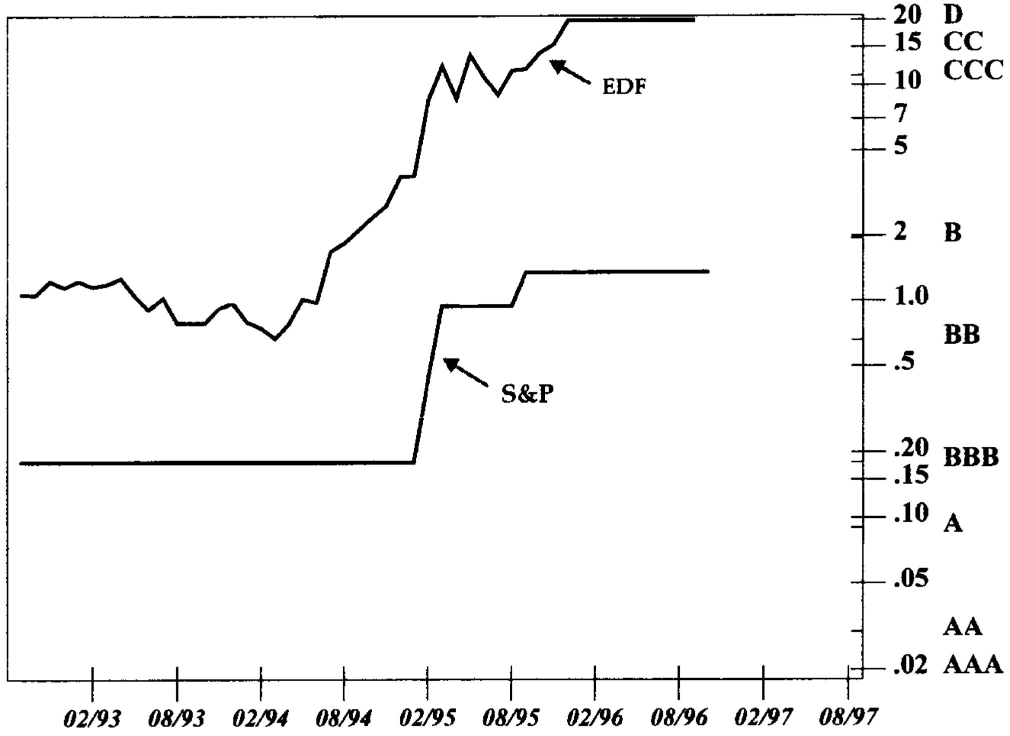
Three steps to derive the actual probabilities of default:

1. Estimation of the market value and volatility of the firm asset value.

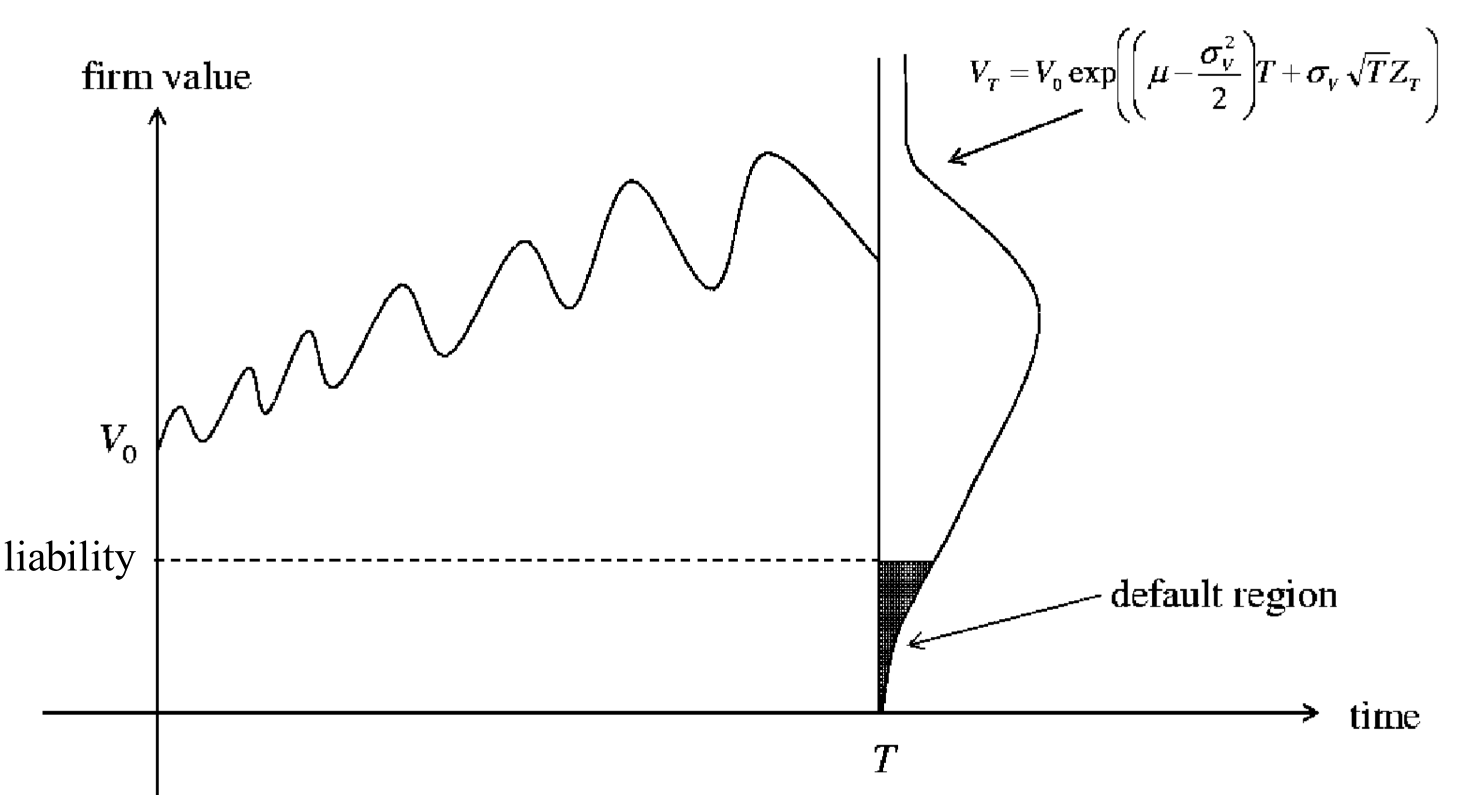
2. Calculation of the distance to default, an index measure of default risk.

3. Scaling of the distance to default to actual probabilities of default using a default database.

- Changes in EDF tend to anticipate at least one year earlier than the downgrading of the issuer by rating agencies like Moodys and S & Ps



Distribution of terminal firm value at maturity of debt



- According to KMV′s empirical studies, log-asset returns confirm quite well to a normal distribution, and σ_V stays relatively constant.

- From the sample of several hundred companies, firms default when the asset value reaches a level somewhere between the value of total liabilities and the value of the short-term debt.

Distance to default

Default point, $d^* =$ short-term debt $+ \frac{1}{2} \times$ long-term debt. Why $\frac{1}{2}$? Why not!

From $V_T = V_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma_V Z_T\right)$, the probability of finishing below D at date T is

$$N\left(-\frac{\ln \frac{V_0}{D} + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma_V \sqrt{T}}\right).$$

Distance to default is defined by

$$d_f = \frac{E(V_T) - d^*}{\widehat{\sigma}_V \sqrt{T}} = \frac{\ln \frac{V_0}{d^*} + \left(\mu - \frac{\widehat{\sigma}_V^2}{2}\right) T}{\widehat{\sigma}_V \sqrt{T}},$$

where V_0 is the current market value of firm, μ is the expected rate of return on firm value and $\widehat{\sigma}_V$ is the annualized firm value volatility.

The probability of default is a function of the firm's capital structure, the volatility of the asset returns and the current asset value.

Estimation of firm value V and volatility of firm value σ_V

- Usually, only the price of equity for most public firms is directly observable. In some cases, part of the debt is directly traded.

- Using option pricing approach:

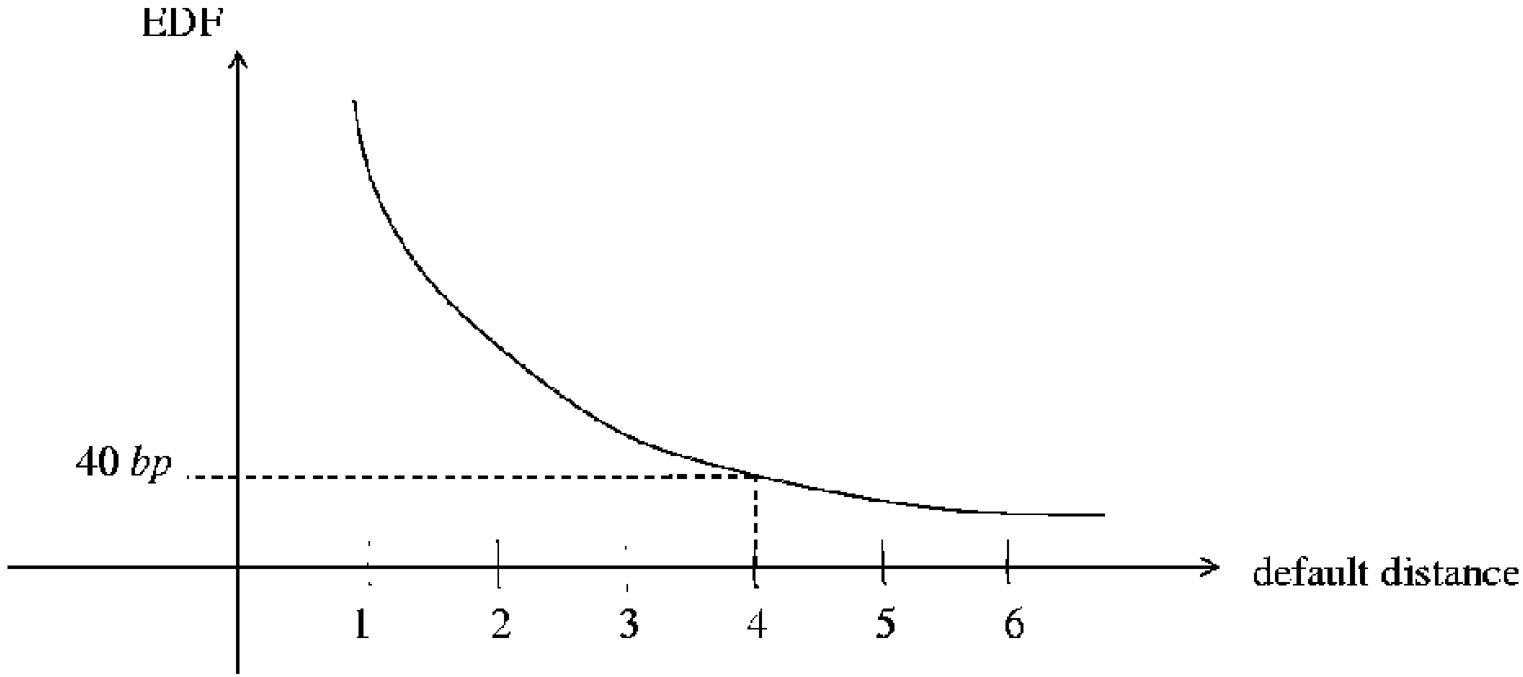
  equity value, $E = f(V, \sigma_V, K, c, r)$

  volatility of equity, $\sigma_E = g(V, \sigma_V, K, c, r)$

  where K denotes the leverage ratio in the capital structure, c is the average coupon paid on the long-term debt, r is the riskfree rate. Actually, the relation between σ_E and σ_V is obtained via the Ito lemma: $E\sigma_E = \dfrac{\partial f}{\partial V} V \sigma_V$.

- Solve for V and σ_V from the above 2 equations.

Probabilities of default from the default distance



Based on historical information on a large sample of firms, for each time horizon, one can estimate the proportion of firms of a given default distance (say, $d_f = 4.0$) which actually defaulted after one year.

Example *Federal Express* (dollars in billion of US$)

	November 1997	*February 1998*
Market capitalization (price $\times$ shares outstanding)	\$7.9	\$7.3
Book liabilities	\$4.7	\$4.9
Market value of assets	\$12.6	\$12.2
Asset volatility	15%	17%
Default point	\$3.4	\$3.5
Default distance	$\dfrac{12.6 - 3.4}{0.15}$	$\dfrac{12.2 - 3.5}{0.17}$
EDF	$0.06\%(6bp) = AA^-$	$0.11\%(11bp) = A^-$

The causes of changes for an EDF are due to variations in the *stock price*, *debt level* (leverage ratio), and *asset volatility*.

Weaknesses of the KMV approach

- It requires some *subjective estimation* of the input parameters.
- It is difficult to construct theoretical EDF's without the *assumption of normality* of asset returns.
- *Private firms' EDFs* can be calculated only by using some compapability analysis based on accounting data.
- It does not *distinguish* among different types of long-term bonds according to their seniority, collateral, covenants or convertibility.