## 2. Default correlation

Correlation of defaults of a pair of risky assets
Consider two obligors $A$ and $B$ and a fixed time horizon $T$.

$$
p_{A}=\text { probability of default of } A \text { before } T
$$

$p_{B}=$ probability of default of $B$ before $T$
$p_{A B}=$ joint default probability that $A$ and $B$ default before $T$
$p_{A \mid B}=$ probability that $A$ defaults before $T$, given that $B$ has defaulted before $T$

$$
p_{A \mid B}=\frac{p_{A B}}{p_{B}}, \quad p_{B \mid A}=\frac{p_{A B}}{p_{A}}
$$

$\rho_{A B}=$ linear correlation coefficient between default events

$$
=\frac{E\left[\mathbf{1}_{\{A\}} \mathbf{1}_{\{B\}}\right]-E\left[\mathbf{1}_{\{A\}}\right] E\left[\mathbf{1}_{\{B\}}\right]}{\sigma_{\mathbf{1}_{\{A\}}} \sigma_{\mathbf{1}_{\{B\}}}}
$$

$$
=\frac{p_{A B}-p_{A} p_{B}}{\sqrt{p_{A}\left(1-p_{A}\right) p_{B}\left(1-p_{B}\right)}}
$$

where $\mathbf{1}_{\{A\}}= \begin{cases}1 & A \text { defaults before } T \\ 0 & \text { otherwise }\end{cases}$

Since default probabilities are very small, the correlation $\rho_{A B}$ can have a much larger effect on the joint risk of a position

$$
\begin{aligned}
p_{A B} & =p_{A} p_{B}+\rho_{A B} \sqrt{p_{A}\left(1-p_{A}\right) p_{B}\left(1-p_{B}\right)} \\
p_{A \mid B} & =p_{A}+\rho_{A B} \sqrt{\frac{p_{A}}{p_{B}}\left(1-p_{A}\right)\left(1-p_{B}\right)} \quad \text { and } \\
p_{B \mid A} & =p_{B}+\rho_{A B} \sqrt{\frac{p_{B}}{p_{A}}\left(1-p_{A}\right)\left(1-p_{B}\right)} .
\end{aligned}
$$

Assume $\rho_{A B}=\rho$ which is not small but $p_{A}=p_{B}=p \ll 1$, then

$$
\begin{aligned}
p_{A B} & \approx p^{2}+\rho p \approx \rho p \\
p_{A \mid B} & \approx \rho
\end{aligned}
$$

The joint default probability and the conditional default probability are dominated by the correlation coefficient $\rho$.

- When there are 2 obligors, we can compute the probabilities of all elementary events by using the linear correlation coefficient.
- With 3 obligors, we have 8 elementary events but only 7 restrictions ( 3 individual probabilities, 3 correlations and sum of probabilities). The probability of the joint default of all 3 obligors is not determined by the 3 pairs of correlation.
- For $N$ obligors, we have $N(N-1) / 2$ correlations, $N$ individual default probabilities. Yet we have $2^{N}$ possible joint default events. The correlation matrix only gives the bivariate marginal distributions, while the full distribution remains undetermined.
- For any pair of random variables $X$ and $Y$

$$
\operatorname{cov}(1-X, 1-Y)=\operatorname{cov}(X, Y)
$$

so that the linear correlation coefficient between survival events is the same as that between default events. Hence, it is unable to capture the fact that risky assets exhibit greater tendency to crash together than to boom together.

Price bounds for first-to-default (FtD) swaps under low default correlation

$$
\left.\begin{array}{lll}
\text { fee on CDS on } \\
\text { worst credit }
\end{array} \quad \begin{array}{l}
\text { fee on FtD } \\
\text { swap }
\end{array} \quad \begin{array}{l}
\text { portfolio of } \\
\text { CDSs on all } \\
\text { Credits }
\end{array}\right] \begin{aligned}
& \bar{s}_{A}+\bar{s}_{B}+\bar{s}_{C}
\end{aligned}
$$

With low default probabilities and low default correlation

$$
\bar{s}^{\mathrm{FtD}} \approx \bar{s}_{A}+\bar{s}_{B}+\bar{s}_{C} .
$$

To see this, the probability of at least one default is

$$
\begin{aligned}
p & =1-\left(1-p_{A}\right)\left(1-p_{B}\right)\left(1-p_{C}\right) \\
& =p_{A}+p_{B}+p_{C}-\left(p_{A} p_{B}+p_{A} p_{C}+p_{B} p_{C}\right)+p_{A} p_{B} p_{C}
\end{aligned}
$$

so that

$$
p \lesssim p_{A}+p_{B}+p_{C} \quad \text { for small } \quad p_{A}, p_{B} \text { and } p_{C}
$$

## Counterparty risk in CDS

Before the Fall 1997 crisis, several Korean banks were willing to offer credit default protection on other Korean firms.


* Higher geographical risks lead to higher default correlations.

Advice: Go for a European bank to buy the protection.

## Nature of counterparty risk

1. Replacement cost

- If the Protection Seller defaults prior to the Reference Entry, then the Protection Buyer renews the CDS with a new counterparty.
- Supposing that the default risks of the Protection Seller and Reference Entity are positively correlated, there will be an increase in the swap rate in the new CDS.

2. Settlement risk

- The Protection Seller defaults during the settlement period after the default of the Reference Entity.

Funding cost arbitrage - Credit default swap


In order that the funding cost arbitrage works, the funding cost of the default protection seller must be higher than that of the default protection buyer.

The combined risk faced by the Protection Buyer:

- default of the BBB-rated bond
- default of the Protection Seller on the contingent payment

The AAA-rated Protection Buyer creates a synthetic AA-asset with a coupon rate of $\mathrm{LIBOR}+90 \mathrm{bps}-50 \mathrm{bps}=\mathrm{LIBOR}+40 \mathrm{bps}$.

This is better than LIBOR $+30 b p s$, which is the coupon rate of a AA-asset (net gains of 10bps).

For the A-rated Protection Seller, it gains synthetic access to a BBB-rated asset with earning of net spread of

$$
50 \mathrm{bps}-[\underbrace{(\text { LIBOR }+90 \mathrm{bps})}_{\begin{array}{c}
\text { the A-rated Protection Seller earns 40bps } \\
\text { if it owns the BB asset directly }
\end{array}}]=10 \mathrm{bps}
$$

## Need for theoretical models of default correlations

Models for default correlation must be able to explain and predict default correlations from few and more fundamental variables than a simple full description of the joint default distribution.

Several possible sources of data, none of which is perfect.

- Historically observed joint rating and default events
- Credit spreads
- Equity correlations

1. We simply do not have direct data on default dependencies.
2. Specification of full joint default probabilities is simply too complex. There are $2^{N}$ joint default events for $N$ obligors.

## Poisson model of default

Suppose default of obligor $i$ is modeled by a Poisson-distributes random variable: $L_{i} \sim P_{\text {ois }}\left(\lambda_{i}\right)$.

Probability of default $=p_{i}=P_{r}\left[L_{i} \geq 1\right]$

$$
=1-e^{-\lambda_{i}} \approx \lambda_{i} \text { for small } \lambda_{i}
$$

## Interacting intensities

Two-firm contagion model

$$
\begin{aligned}
& \lambda_{t}^{A}=a_{0}+a_{1} \mathbf{1}_{\left\{\tau^{B} \leq t\right\}} \\
& \lambda_{t}^{B}=b_{0}+b_{1} \mathbf{1}_{\left\{\tau^{A} \leq t\right\}}
\end{aligned}
$$

The default intensity $\lambda_{t}^{A}\left(\lambda_{t}^{B}\right)$ jumps by the amount $a_{1}\left(b_{1}\right)$ when Firm $B(A)$ defaults.

## Stochastic intensities

Default intensities for different companies follow correlated stochastic processes:

$$
d \lambda_{j}(t)=K[\theta-\lambda(t)] d t+\sigma_{j} \sqrt{\lambda(t)} d Z_{j} \quad d Z_{j} d Z_{k}=\rho_{j k} d t
$$

$\lambda_{j}(t)$ is said to follow the CIR process and $\rho_{j k}$ is the correlation coefficient between the Brownian motions.

## Contagion model

Contagion means that once a firm defaults, it may bring down other firms with it.

Davis-Lo model (2001)
$Y_{i j}$ is an "infection" variable which, when equal to one, implies that default of Firm $i$ immediately triggers default of Firm $j$.

Assume $X_{i}, Y_{i j}$ are independent Bernuolli variables with

$$
P\left[X_{i}=1\right]=p \quad \text { and } \quad P\left[Y_{i j}=1\right]=q
$$

Define $Z_{i}$ be the default indicator of Firm $i$

$$
Z_{i}=X_{i}+\left(1-X_{i}\right)\left[1-\prod_{j \neq i}\left(1-X_{j} Y_{j i}\right)\right]
$$

Note that $Z_{i}$ equals one either when there is a direct default of firm $i$ or if there is no direct default and $\prod_{j \neq i}\left(1-X_{j} Y_{j i}\right)=0$. The latter case occurs when at least one of the factor $X_{j} Y_{j i}$ is 1 , which happens when firm $j$ defaults and infects firm $i$.

Define $D_{n}=Z_{1}+\cdots+Z_{n}$, Davis and Lo (2001) find that

$$
\begin{aligned}
E\left[D_{n}\right] & =n\left[1-(1-p)(1-p q)^{n-1}\right] \\
\operatorname{var}\left(D_{n}\right) & =n(n-1) \beta_{n}^{p q}-\left(E\left[D_{n}\right]\right)^{2}
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_{n}^{p q}= & p^{2}+2 p(1-p)\left[1-(1-q)(1-p q)^{n-2}\right] \\
& +(1-p)^{2}\left[1-2(1-p q)^{n-2}+\left(1-2 p q+p q^{2}\right)^{n-2}\right] \\
& \operatorname{cov}\left(Z_{i}, Z_{j}\right)=\beta_{n}^{p q}-\operatorname{var}\left(D_{n} / n\right)^{2}
\end{aligned}
$$

## Moody's binomial expansion technique

- Diversity score, weighted average rating factor and binomial expansion technique.
- Generate the loss distribution.

To build a hypothetical pool of uncorrelated and homogeneous assets that mimic the default behaviors of the original pool of correlated and inhomogeneous assets.

Additional assumptions

- Every instrument in the comparison portfolio can be uniquely assigned to one industry group.
- Two instruments in the comparison portfolio have positive correlation if and only if they belong to the same industry group.


## Moody's diversity score

The diversity score of a given pool of participations is the number $n$ of bonds in a idealized comparison portfolio that meets the following criteria:

- Comparison portfolio and collateral pool have the same face value.
- Bonds in the comparison portfolio have equal face values.
- Comparison bonds are equally likely to default, and their default is independent.
- Comparison bonds are of the same average default probability as the participations of the collateral pool.
- Comparison portfolio has, according to some measure of risk, the same total risk as does the collateral pool.

Moody's Diversity Scores for Firms within an Industry

|  |  | $P\left(d_{i} \mid d_{j}\right)$ |  |  | $\left(d_{i}, d_{j}\right)$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Firms in industry | Diversity score | $(q=0.5)$ | $(q=0.05)$ |  | $(q=0.5)$ | $(q=0.05)$ |
| 1 | 1.00 |  |  |  |  |  |
| 2 | 1.50 | 0.78 | 0.48 |  | 0.56 | 0.45 |
| 3 | 2.00 | 0.71 | 0.37 |  | 0.42 | 0.34 |
| 4 | 2.33 | 0.70 | 0.36 |  | 0.40 | 0.32 |
| 5 | 2.67 | 0.68 | 0.33 |  | 0.36 | 0.30 |
| 6 | 3.00 | 0.67 | 0.31 |  | 0.33 | 0.27 |
| 7 | 3.25 | 0.66 | 0.30 |  | 0.32 | 0.26 |
| 8 | 3.50 | 0.65 | 0.29 |  | 0.31 | 0.25 |
| 9 | 3.75 | 0.65 | 0.27 |  | 0.29 | 0.24 |
| 10 | 4.00 | 0.64 | 0.26 | 0.28 | 0.23 |  |
| $>10$ | Evaluated on a case-by-case basis |  |  |  |  |  |

Source: The assumed diversity score is that assigned by Moody's, as tabulated by Schorin and Weinreich (1998).
Note: Here, $q$ is the default probability of an arbitrary name, while $d_{i}$ is the event of default by name $i$. The conditional probability and default correlation calculations assume symmetry.

## Binomial probability formula

Once the "average" default probability p is known, then the probability of $k$ defaults out of $n$ bonds will be given by the probability formula of $k$ successes out of $n$ independent trials.

$$
P(k \text { defaults })=\frac{n!}{(n-k)!k!} p^{k}(1-p)^{n-k}
$$

where default probability

$$
p=\frac{\sum_{n=1}^{n} p_{i} F_{i}}{\sum_{i=1}^{n} F_{i}}
$$

## Binomial mixture model

Mixture distribution randomizes the default probability of the binomial model to induce dependence, thus mimicking a situation where a common background variable affects a collection of firms. The default events of the firms are then conditionally independent given the mixture variable.

## Binomial distribution

Suppose $X$ is binomially distributed ( $n, p$ ), then

$$
E[X]=n p \quad \text { and } \quad \operatorname{var}(X)=n p(1-p)
$$

We randomize the default parameter $p$. Recall the following relationships for random variables $X$ and $Y$ defined on the same probability space

$$
E[X]=E[E[X \mid Y]] \quad \text { and } \quad \operatorname{var}(X)=\operatorname{var}(E[X \mid Y])+E[\operatorname{var}(X \mid Y)]
$$

Suppose we have a collection of $n$ firms, $X_{i}=D_{i}(T)$ is the default indicator of firm $i$. Assume that $\widetilde{p}$ is a random variable which is independent of all the $X_{i}$. Assume that $\widetilde{p}$ takes on values in $[0,1]$. Conditional on $\widetilde{p}, X_{1}, \cdots, X_{n}$ are independent and each has default probability $\widetilde{p}$. Let $\bar{p}$ denote the mean of $\widetilde{p}$, where

$$
\bar{p}=E[\widetilde{p}]=\int_{0}^{1} p f(p) d p
$$

We have

$$
E\left[X_{i}\right]=\bar{p} \quad \text { and } \quad \operatorname{var}\left(X_{i}\right)=\bar{p}(1-\bar{p})
$$

and

$$
\operatorname{cov}\left(X_{i}, X_{j}\right)=E\left[\tilde{p}^{2}\right]-\bar{p}^{2}, \quad i \neq j
$$

(i) When $\widetilde{p}$ is a constant, we have zero covariance.
(ii) By Jensen's inequality, $\operatorname{cov}\left(X_{i}, X_{j}\right) \geq 0$.
(iii) Default event correlation

$$
\rho\left(X_{i}, X_{j}\right)=\frac{E\left[\widetilde{p}^{2}\right]-\bar{p}^{2}}{\bar{p}(1-\bar{p})}
$$

Define $D_{n}=\sum_{i=1}^{n} X_{i}$, which is the total number of defaults; then

$$
E\left[D_{n}\right]=n \bar{p} \quad \text { and } \quad \operatorname{var}\left(D_{n}\right)=n \bar{p}(1-\bar{p})+n(n-1)\left(E\left[\tilde{p}^{2}\right]-E[\widehat{p}]^{2}\right)
$$

(i) When $\widetilde{p}=\bar{p}$, corresponding no randomness, $\operatorname{var}\left(D_{n}\right)=n \bar{p}(1-\bar{p})$, like usual binomial distribution.
(ii) When $\widetilde{p}=1$ with prob $\bar{p}$ and zero otherwise, then $\operatorname{var}\left(D_{n}\right)=$ $n^{2} \bar{p}(1-\bar{p})$, corresponding to perfect correlation between all default events.
(iii) One can obtain any default correlation in [ 0,1 ]; correlation of default events depends only on the first and second moments of $f$. However, the distribution of $D_{n}$ can be quite different.
(iv) $\operatorname{var}\left(\frac{D_{n}}{n}\right)=\frac{\bar{p}(1-\bar{p})}{n}+\frac{n(n-1)}{n^{2}} \operatorname{var}(\tilde{p}) \longrightarrow \operatorname{var}(\tilde{p})$ as $n \rightarrow \infty$, that is, when considering the fractional loss for $n$ large, the only remaining variance is that of the distribution of $\widetilde{p}$.

Copula approach for modeling default dependency
Two aspects of modeling the default times of several obligors

1. Default dynamics of a single obligor.
2. Model the dependence structure of defaults between the obligors.

Question How to specify a joint distribution of survival times, with given marginal distributions?

- Knowing the joint distribution of random variables allows us to derive the marginal distributions and the correlation structure among the random variables but not vice versa.
- A copula function links univariate marginals to their full multivariate distribution.


## Proposition

If an one-dimensional continuous random variable $X$ has distribution function $F$, that is, $F(x)=P[X \leq x]$, then the distribution of the random variable $U=F(X)$ is a uniform distribution on $[0,1]$.

Remark
To simulate an outcome of $X$, one may simulate an outcome $u$ from a uniform distribution then let the outcome of $X$ be $x=F^{-1}(u)$.

## Multi-variate distribution function

$$
F_{X_{1}, X_{2}, \cdots, X_{n}}\left(x_{1}, x_{2}, \cdots, x_{n}\right)=P\left[X_{1} \leq x_{1}, X_{2} \leq x_{2}, \cdots, X_{n} \leq x_{n}\right]
$$

- It is a non-decreasing, right continuous function which maps a subset of the real numbers into the unit interval $[0,1]$.
- Monotonicity property for vector $\mathbf{a}$ and $\mathbf{b}$

$$
\mathbf{a}<\mathbf{b} \quad \Rightarrow \quad F(\mathbf{b})-F(\mathbf{a}) \geq 0
$$

$\mathbf{a}<\mathbf{b}$ means $\mathbf{b}-\mathbf{a}$ is a vector with non-negative entries and at least one strictly positive entry.

## Definition of a copula function

A function $C:[0,1]^{n} \rightarrow[0,1]$ is a copula if
(a) There are random variables $U_{1}, U_{2}, \cdots, U_{n}$ taking values in $[0,1]$ such that $C$ is their distribution function.
(b) $C$ has uniform marginal distributions; for all $i \leq n, u_{i} \in[0,1]$

$$
C\left(1, \cdots, 1, u_{i}, 1, \cdots, 1\right)=u_{i}
$$

In the analysis of dependency with copula function, the joint distribution can be separated into two parts, namely, the marginal distribution functions of the random variables (marginals) and the dependence structure between the random variables which is described by the copula function.

## Construction of multi-variate distribution function

Given univariate marginal distribution functions $F_{1}\left(x_{1}\right), F_{2}\left(x_{2}\right), \cdots, F_{n}\left(x_{n}\right)$, the function

$$
C\left(F_{1}\left(x_{1}\right), F_{2}\left(x_{2}\right), \cdots, F_{n}\left(x_{n}\right)\right)=F\left(x_{1}, x_{2}, \cdots, x_{n}\right)
$$

which is defined using a copula function $C$, results in a multivariate distribution function with univariate marginal distributions specified as $F_{1}\left(x_{1}\right), F_{2}\left(x_{2}\right), \cdots, F_{n}\left(x_{n}\right)$.

- Any multi-variate distribution function $F$ can be written in the form of a copula function.


## Theorem

If $F\left(x_{1}, x_{2}, \cdots, x_{n}\right)$ is a joint multi-variate distribution function with univariate marginal distribution functions $F_{1}\left(x_{1}\right), \cdots F_{n}\left(x_{n}\right)$, then there exists a copula function $C\left(u_{1}, u_{2}, \cdots, u_{n}\right)$ such that

$$
F\left(x_{1}, x_{2}, \cdots, x_{n}\right)=C\left(F_{1}\left(x_{1}\right), F_{2}\left(x_{2}\right), \cdots, F_{n}\left(x_{n}\right)\right) .
$$

If each $F_{i}$ is continuous, then $C$ is unique.
Going through all copula functions gives us all the possible types of dependence structures that are compatible with the given onedimensional marginal distributions.


Generating correlated default times via the copula approach (illustrative: here, $F_{1}=F_{2}$ ).

Bivariate normal copula function

$$
C(u, v)=N_{2}\left(N^{-1}(u), N^{-1}(v) ; \gamma\right), \quad-1 \leq \gamma \leq 1 .
$$

Suppose we use a bivariate normal copula function with a correlation parameter $\gamma$, and denote the default times for $A$ and $B$ as $T_{A}$ and $T_{B}$. The joint default probability is given by

$$
\begin{equation*}
P\left[T_{A}<1, T_{B}<1\right]=N_{2}\left(N^{-1}\left(F_{A}(1)\right), N^{-1}\left(F_{B}(1)\right), \gamma\right) \tag{B}
\end{equation*}
$$

where $F_{A}$ and $F_{B}$ are the distribution functions for the default times $T_{A}$ and $T_{B}$.

We observe that

$$
q_{i}=P\left[T_{i}<1\right]=F_{i}(1) \quad \text { and } \quad Z_{i}=N^{-1}\left(q_{i}\right) \quad \text { for } \quad i=A, B
$$

Eqs. ( $A$ ) and (B) are equivalent if we have $\rho=\gamma$.
Note that this correlation parameter is not the correlation coefficient between the two default times.

## Simulation of default times of a basket of obligors

Assume that for each credit $i$ in the portfolio, we have constructed a credit curve or a hazard rate function for its default time $T_{i}$. Let $F_{i}(t)$ denote the distribution function of $T_{i}$.

Using a copula function $C$, we obtain the joint distribution of the default times

$$
F\left(t_{1}, t_{2}, \cdots, t_{n}\right)=C\left(F\left(t_{1}\right), F_{2}\left(t_{2}\right), \cdots, F_{n}\left(t_{n}\right)\right)
$$

For example, suppose we use the normal copula function, we have

$$
F\left(t_{1}, t_{2}, \cdots, t_{n}\right)=N_{n}\left(N^{-1}\left(F_{1}\left(t_{1}\right)\right), N^{-1}\left(F_{2}\left(t_{2}\right)\right), \cdots, N^{-1}\left(F_{n}\left(t_{n}\right)\right)\right)
$$

To simulate correlated default times, we introduce

$$
Y_{1}=N^{-1}\left(F_{1}\left(T_{1}\right)\right), Y_{2}=N^{-1}\left(F_{2}\left(T_{2}\right)\right), \cdots, Y_{n}=N^{-1}\left(F_{n}\left(T_{n}\right)\right)
$$

There is a one-to-one mapping between $Y$ and $T$. This provides the linkage between the random default time $T$ and the credit index $Y$ (modeled as standard normal random variable).

Simulation scheme

- Simulate $Y_{1}, Y_{2}, \cdots, Y_{n}$ from an $n$-dimensional normal distribution with correlation coefficient matrix $\Sigma$.
- Obtain $T_{1}, T_{2}, \cdots, T_{n}$ using $T_{i}=F_{i}^{-1}\left(N\left(Y_{i}\right)\right), \quad i=1,2, \cdots, n$.

With each simulation run, we generate the default times for all the credits in the portfolio. With this information we can value any credit derivative structure written on the portfolio.

## CreditMetrics

- CreditMetrics uses the normal copula function in its default correlation formula even though it does not use the concept of copula function explicitly.
- CreditMetrics calculates joint default probability of two credits $A$ and $B$ using the following steps:
(i) Let $q_{A}$ and $q_{B}$ denote the one-year default probabilities for $A$ and $B$, respectively. Let $Z_{A}$ and $Z_{B}$ denote the credit index of $A$ and $B$, respectively, both are standard normal random variables.

$$
q_{A}=P\left[Z_{A}<z_{A}\right] \quad \text { and } \quad q_{B}=P\left[Z_{B}<z_{B}\right]
$$

(ii) Let $\rho$ denote the asset correlation, the joint default probability for credit $A$ and $B$ is given by

$$
\begin{equation*}
P\left[Z_{A}<z_{A}, Z_{B}<z_{B}\right]=\int_{-\infty}^{z_{A}} \int_{-\infty}^{z_{B}} n_{2}(x, y ; \rho) d x d y=N_{2}\left(z_{A}, z_{B} ; \rho\right) \tag{A}
\end{equation*}
$$

## Factor copula model

Consider a portfolio of $N$ companies and assume that the marginal risk neutral probabilities of default are known for each company.
$\tau_{i}$ : time of default of company $i$
$Q_{i}(t)$ : cumulative risk neutral probability that company $i$ will default before time $t$
$S_{i}(t)=1-Q_{i}(t)=$ risk neutral survival probability.

To generate a one-factor model for $\tau_{i}$, we define

$$
\begin{equation*}
x_{i}=a_{i} M+\sqrt{1-a_{i}^{2}} Z_{i} \tag{1}
\end{equation*}
$$

where $M$ is the common factor and $Z_{i}$ represents the idiosyncratic risk.

We take $M$ and $Z_{i}$ to be independent zero-mean unit-variance distributions and $-1 \leq a_{i} \leq 1$.

Eq. (1) defines a correlation structure between $x_{i}$ dependent on a common factor $M$. The correlation between $x_{i}$ and $x_{j}$ is $a_{i} a_{j}$.

Let $F_{i}$ denote the cumulative distribution of $x_{i}$. Under the copula model, $x_{i}$ is mapped to $\tau_{i}$ using percentile-to-percentile transformation. The point $x_{i}=x$ is transformed to $\tau_{i}=t$ so that

$$
t=Q_{i}^{-1}\left[F_{i}(x)\right]
$$

Assuming $Z_{i}$ to be identically distributed, we let $H$ denote the cumulative distribution of $Z_{i}$.

$$
P_{r}\left[x_{i}<x \mid M\right]=H\left(\frac{x-a_{i} M}{\sqrt{1-a_{i}^{2}}}\right)
$$

When $x=F_{i}^{-1}\left[Q_{i}(t)\right], P_{r}\left[\tau_{i}<t\right]=P_{r}\left[x_{i}<x\right]$ so that

$$
\operatorname{Pr}\left[\tau_{i}<t \mid M\right]=H\left(\frac{F_{i}^{-1}\left(Q_{i}(t)\right)-a_{i} M}{\sqrt{1-a_{i}^{2}}}\right)
$$

Conditional survival probability that the $i^{\text {th }}$ obligor will survive beyond $T$

$$
S_{i}(T \mid M)=1-H\left(\frac{F_{i}^{-1}\left(Q_{i}(T)\right)-a_{i} M}{\sqrt{1-a_{i}^{2}}}\right)
$$

Extension to multi-factor

$$
\begin{aligned}
& x_{i}=a_{i 1} M_{1}+\cdots+a_{i m} M_{m}+Z_{i} \sqrt{1-a_{i 1}^{2}-a_{i 2}^{2}-\cdots-a_{i m}^{2}} \\
& S_{i}\left(T \mid M_{1}, M_{2}, \cdots, M_{m}\right)=1-H\left(\frac{F_{i}^{-1}\left(Q_{i}(T)\right)-a_{i 1} M_{1}-\cdots-a_{i m} M_{m}}{\sqrt{1-a_{i 1}^{2}-\cdots-a_{i m}^{2}}}\right) .
\end{aligned}
$$

Define $\pi_{T}(k)=$ probability that exactly $k$ defaults before $T$

$$
\begin{aligned}
\pi_{T}\left(0 \mid M_{1}, M_{2}, \cdots, M_{m}\right)= & \prod_{i=1}^{N} S_{i}\left(T \mid M_{1}, M_{2}, \cdots, M_{m}\right) \\
\pi_{T}\left(1 \mid M_{1}, M_{2}, \cdots, M_{m}\right)= & \pi_{T}\left(0 \mid M_{1}, M_{2}, \cdots, M_{m}\right) \\
& \sum_{i=1}^{N} \frac{1-S_{i}\left(T \mid M_{1}, M_{2}, \cdots, M_{m}\right)}{S_{i}\left(T \mid M_{1}, M_{2}, \cdots, M_{m}\right)} .
\end{aligned}
$$

Calculations of $\pi_{T}\left(k \mid M_{1}, M_{2}, \cdots, M_{m}\right)$ for $k \geq 2$ can be relegated to multinomial formula.

Spearman's rho correlation
It is simply the linear correlation of the probability-transformed random variables. It can be expressed in terms of copula $K^{\tau}$ as follows:

$$
\begin{aligned}
\rho^{S}\left(\tau_{1}, \tau_{2}\right) & =\rho\left(F_{1}\left(\tau_{1}\right), F_{2}\left(\tau_{2}\right)\right) \\
& =12 \int_{0}^{1} \int_{0}^{1}\left[K^{\tau}(u, v)-u v\right] d u d v
\end{aligned}
$$

Kendall's tau correlation

$$
\rho^{\tau}\left(\tau_{1}, \tau_{2}\right)=E\left[\operatorname{sign}\left(\tau_{1}-\widetilde{\tau}_{1}\right)\left(\tau_{2}-\widetilde{\tau}_{2}\right)\right]
$$

where $\left(\widetilde{\tau}_{1}, \widetilde{\tau}_{2}\right)$ is an independent copy of $\left(\tau_{1}, \tau_{2}\right)$.

$$
\rho^{\tau}\left(\tau_{1}, \tau_{2}\right)=4 \int_{0}^{1} \int_{0}^{1} K^{\tau}(u, v) d K^{\tau}(u, v)-1
$$

## CreditRisk ${ }^{+}$

- A credit risk model developed by Credit Suisse Financial Products (CSFP).
- A typical representative of the group of Poisson mixture models.
- Gamma distribution

$$
\begin{aligned}
\gamma_{\alpha, \beta}(x) & =\frac{1}{\beta^{\alpha} \Gamma(\alpha)} e^{-x / \beta} x^{\alpha-1}, \quad x \geq 0 \\
E[\Lambda] & =\alpha \beta \quad \text { and } \quad \operatorname{var}(\Lambda)=\alpha \beta^{2}
\end{aligned}
$$

- Sector model

Every sector could be thought of as generated by a single underlying factor. Sectors can be identified with industries, countries or regions, or any other systematic influence on the economic performance of counterparties with a positive weight in this sector.

1. Each sector $s \in\left\{1, \cdots, m_{s}\right\}$ has its own gamma-distributed random intensity

$$
\Lambda^{(s)} \sim \Gamma\left(\alpha_{s}, \beta_{s}\right)
$$

and $\Lambda^{(1)}, \cdots, \Lambda^{\left(m_{S}\right)}$ are assumed to be independent.
2. Every obligor $i$ admits a breakdown into sector weights $w_{i s} \geq 0$ with

$$
\sum_{s=1}^{m_{S}} w_{i s}=1
$$

3. The risk of sector $s$ is captured by two parameters
(i) mean default intensity of the sector

$$
\lambda_{(s)}=E\left[\wedge^{(s)}\right]=\alpha_{s} \beta_{s}
$$

(ii) default intensity's volatility

$$
\sigma_{(s)}^{2}=\operatorname{var}\left(\Lambda^{(s)}\right)=\alpha_{s} \beta_{s}^{2}
$$

Every obligor $i$ admits a random default intensity $\wedge_{i}$ with mean value

$$
E\left[\wedge_{i}\right]=\lambda_{i}
$$

which could be calibrated to the obligor's one-year default probability.

The sector parametrization of $\wedge_{i}$

$$
\wedge_{i}=\sum_{s=1}^{m_{S}} w_{i s} \lambda_{i} \frac{\wedge^{(s)}}{\lambda_{(s)}}, \quad i=1,2, \cdots, m
$$

Two obligors are correlated iff there is at least one sector such that both obligors have a positive sector weight with respect to this sector - admit a common source of systematic default risk.

Mixed Poisson random variable $L_{s}^{\prime}$
The default risk of obligor $i$ is modeled by a mixed Poisson random variable $L_{s}^{\prime}$ with random intensity $\wedge_{i}$.

Any conditional default intensity of obligor $i$ arising from realizations $\theta_{1}, \cdots, \theta_{m_{S}}$ of the sector's default intensities $\wedge^{(1)}, \cdots, \wedge^{\left(m_{S}\right)}$ generates a conditional one-year default probability $p_{i}\left(\theta_{1}, \cdots, \theta_{m_{S}}\right)$ of obligor $i$ by

$$
\begin{aligned}
p_{i}\left(\theta_{1}, \cdots, \theta_{m_{S}}\right) & =P\left[L_{i}^{\prime} \geq 1 \mid \wedge^{(1)}=\theta_{1}, \cdots, \Lambda^{\left(m_{S}\right)}=\theta_{m_{S}}\right] \\
& =1-e^{-\lambda_{i} \sum_{s=1}^{m_{S}} w_{i s} \theta_{s} / \lambda_{(s)}}
\end{aligned}
$$

## Summary

A good portfolio credit risk model should have the following properties

- Default dependence - produce default correlations of a realistic magnitude.
- Estimation - number of parameters should be limited.
- Timing risk - producing "clusters" of defaults in time, several defaults that occur close to each other
- Calibration (i) Individual term structures of default probabilities
(ii) Joint defaults and correlation information
- Implementation - existence of a variable implementation mechanism, say Monte Carlo simulation method

